A LEVEL MATHS 9709 · PAPER 3 PURE MATHEMATICS 3 · 3.1 · ~11 MIN READ

Algebra: modulus, polynomials and partial fractions

The modulus function and its equations and inequalities, polynomial division, the factor and remainder theorems, partial fractions for the three syllabus forms, and the binomial expansion for any rational power with its range of validity.

Modulus graphs | Modulus equations & inequalities | Polynomial division | Factor theorem | Remainder theorem | Partial fractions | Binomial expansion (rational n) | Range of validity

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Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded

highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **|x| is the distance from 0.** Solve |a| = |b| by squaring ($a^2 = b^2$), and |x − a| < b as a − b < x < a + b.
- **Remainder theorem:** dividing p(x) by (ax + b) leaves remainder p(−b/a). **Factor theorem:** (ax + b) is a factor exactly when p(−b/a) = 0.
- **Partial fractions** split a proper rational function into simpler fractions you can integrate or expand: one term per linear factor, two for a repeated factor, (Bx + C) over an irreducible quadratic.
- **$(1 + x)^n$ for rational n** is an infinite series, valid only for |x| < 1. Adapt it by taking out a factor so the bracket starts with 1.

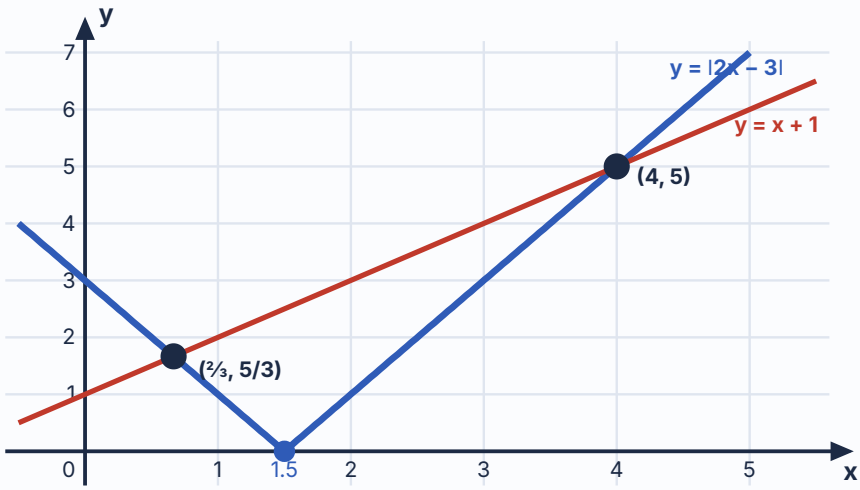
BY THE END YOU CAN

- Sketch y = |ax + b| and solve modulus equations and inequalities.
- Divide a polynomial by a linear or quadratic divisor.
- Use the factor and remainder theorems, including factors (ax + b).
- Split rational functions into partial fractions of the three syllabus forms.
- Expand $(1 + x)^n$ for rational n, adapt it, and state the set of x for which it is valid.

1 The modulus function

Graphs and equations

|x| — the size of x without its sign: |x| = x if x ≥ 0, and |x| = −x if x < 0.



The V reflects the part of y = 2x − 3 below the axis. The line meets each arm once.

y = |2x − 3| is the line y = 2x − 3 with the part below the x-axis reflected upwards.

- **Sketching y = |ax + b|:** a V shape with its vertex on the x-axis at x = −b/a, y-intercept |b|. Paper 3 only needs linear expressions inside the modulus.
- **|a| = |b| ⇔ a² = b².** Squaring removes both moduli without needing cases.
- **|a| = b (b an expression):** solve a = b and a = −b, then **check** each answer, since b must be non-negative.

WORKED EXAMPLE

Solve |3x − 2| = |2x + 7|.

1. square both sides: $9x^2 - 12x + 4 = 4x^2 + 28x + 49$ M1
2. $5x^2 - 40x - 45 = 0 \Rightarrow x^2 - 8x - 9 = 0 \Rightarrow (x - 9)(x + 1) = 0$ A1
3. **x = 9 or x = −1** A1
4. **check: |25| = |25| and |−5| = |5| ✓**

WORKED EXAMPLE

Solve |2x − 3| = x + 1.

1. $2x - 3 = x + 1 \Rightarrow x = 4$; $-(2x - 3) = x + 1 \Rightarrow x = \frac{2}{3}$ M1 A1
2. **both give x + 1 > 0, so both are valid: x = 4 or x = ⅔** A1

Inequalities

$$|x - a| < b \Leftrightarrow a - b < x < a + b$$ LEARN IT

and $|x - a| > b \Leftrightarrow x < a - b$ or $x > a + b$

- **Two moduli:** |f(x)| < |g(x)| ⇔ f(x)² < g(x)². Solve the quadratic inequality.
- **One modulus and a line:** find the critical values (as for the equation), then use a sketch to choose the region.

WORKED EXAMPLE

Solve $2x + 5 < |x + 1|$.

1. for $x \geq -1$: $2x + 5 < x + 1 \Rightarrow x < -4$, which contradicts $x \geq -1$: no solutions M1
2. for $x < -1$: $2x + 5 < -x - 1 \Rightarrow x < -2$ M1
3. **$x < -2$** A1 **(check: $x = -3$ gives $-1 < 2$ ✓)**

COMMON MISTAKE

Squaring when only one side has a modulus, without checking. Squaring can introduce extra solutions; test each answer in the original equation.

2 Polynomials

Division

- **Long division** works like number division. Line up powers of x, including any missing powers with zero coefficients.
- **Result:** p(x) = (divisor) × (quotient) + remainder, with the remainder of lower degree than the divisor.

WORKED EXAMPLE

Divide $2x^3 + x^2 - 13x + 6$ by $(x - 2)$, and hence factorise fully.

1. $2x^3 \div x = 2x^2$; subtract $2x^2(x - 2) = 2x^3 - 4x^2$ to leave $5x^2 - 13x$ M1
2. $5x^2 \div x = 5x$; subtract $5x(x - 2)$ to leave $-3x + 6$; $-3 \times (x - 2)$ leaves 0 A1
3. quotient **$2x^2 + 5x - 3$**, remainder **0** A1
4. **so $(x - 2)(2x - 1)(x + 3)$** A1

Factor and remainder theorems

$$p(x) \div (ax + b) \text{ leaves remainder } p(-b/a)$$ LEARN IT

remainder theorem; the factor theorem is the case remainder = 0

- **Why:** $p(x) = (ax + b)q(x) + R$. Put $x = -b/a$ and the first term vanishes, leaving R.

WORKED EXAMPLE

$p(x) = x^3 + ax^2 + bx - 6$. $(x - 1)$ is a factor, and dividing by $(x + 2)$ leaves -12. Find a and b.

1. $p(1) = 0 \Rightarrow 1 + a + b - 6 = 0 \Rightarrow a + b = 5$ M1
2. $p(-2) = -12 \Rightarrow -8 + 4a - 2b - 6 = -12 \Rightarrow 2a - b = 1$ M1
3. **$a = 2$, $b = 3$** A1 A1

WORKED EXAMPLE

Show that $(2x + 1)$ is a factor of $2x^3 + 3x^2 - 11x - 6$, and solve the equation = 0.

1. $p(-½) = 2(-⅛) + 3(¼) - 11(-½) - 6 = -¼ + ¾ + 5½ - 6 = 0$, so $(2x + 1)$ is a factor M1 A1
2. $2x^3 + 3x^2 - 11x - 6 = (2x + 1)(x^2 + x - 6) = (2x + 1)(x + 3)(x - 2)$ M1
3. **$x = -½, -3, 2$** A1

COMMON MISTAKE

Solving a cubic on a calculator. The 2024 Paper 3 report: calculators may solve quadratics, but cubics and quartics must be factorised with working shown.

3 Partial fractions

The three forms

Denominator	Form to use
$(ax + b)(cx + d)(ex + f)$	$A/(ax + b) + B/(cx + d) + C/(ex + f)$
$(ax + b)(cx + d)^2$	$A/(ax + b) + B/(cx + d) + C/(cx + d)^2$
$(ax + b)(cx^2 + d)$	$A/(ax + b) + (Bx + C)/(cx^2 + d)$

- **The numerator must have lower degree than the denominator** (in this syllabus it always will).
- **Finding constants:** multiply through by the denominator, then substitute convenient values of x (the roots of each factor), and compare coefficients for any constant left.

WORKED EXAMPLE

Express $(5x + 1)/[(x - 1)(x + 2)]$ in partial fractions.

1. $5x + 1 \equiv A(x + 2) + B(x - 1)$ M1
2. $x = 1$: $6 = 3A \Rightarrow A = 2$; $x = -2$: $-9 = -3B \Rightarrow B = 3$ A1
3. **$2/(x - 1) + 3/(x + 2)$** A1

WORKED EXAMPLE

Express $4/[(x + 1)(x - 1)^2]$ in partial fractions.

1. $4 \equiv A(x - 1)^2 + B(x + 1)(x - 1) + C(x + 1)$ M1
2. $x = 1$: $4 = 2C \Rightarrow C = 2$; $x = -1$: $4 = 4A \Rightarrow A = 1$ A1
3. x^2 terms: $0 = A + B \Rightarrow B = -1$ A1
4. **$1/(x + 1) - 1/(x - 1) + 2/(x - 1)^2$**

WORKED EXAMPLE

Express $(3x^2 + x + 4)/[(x + 1)(x^2 + 1)]$ in partial fractions.

1. $3x^2 + x + 4 \equiv A(x^2 + 1) + (Bx + C)(x + 1)$ M1
2. $x = -1$: $6 = 2A \Rightarrow A = 3$; x^2: $3 = A + B \Rightarrow B = 0$; constants: $4 = A + C \Rightarrow C = 1$ A1 A1
3. **$3/(x + 1) + 1/(x^2 + 1)$**

COMMON MISTAKE

Using $A/(x - 1) + B/(x - 1)$ for a repeated factor (it collapses to one fraction), or forgetting the Bx term over a quadratic. Both were common in 2024.

4 The binomial expansion for rational n

The series

$$(1 + x)^n = 1 + nx + [n(n - 1)/2!]x^2 + [n(n - 1)(n - 2)/3!]x^3 + \dots , |x| < 1$$ FORMULA LIST (MF19)

for any rational n; an infinite series unless n is a positive integer

- **Adapting:** make the bracket start with 1. $(a + bx)^n = a^n(1 + bx/a)^n$, valid for $|bx/a| < 1$, i.e. $|x| < |a/b|$.
- **Brackets matter:** in $(1 - 4x)^{1/2}$, the "x" of the formula is $(-4x)$, so x^2 becomes $16x^2$ and x^3 becomes $-64x^3$.

WORKED EXAMPLE

Expand $\sqrt{(1 - 4x)}$ in ascending powers of x up to x^3, and state the values of x for which it is valid.

1. $(1 + (-4x))^{1/2} = 1 + \frac{1}{2}(-4x) + [(\frac{1}{2})(-\frac{1}{2})/2](-4x)^2 + [(\frac{1}{2})(-\frac{1}{2})(-3/2)/6](-4x)^3$ M1
2. $= \mathbf{1 - 2x - 2x^2 - 4x^3}$ A1 A1
3. **valid for $|4x| < 1$, i.e. $|x| < \frac{1}{4}$** B1

WORKED EXAMPLE

Expand $(2 + x)^{-2}$ up to x^3.

1. $(2 + x)^{-2} = 2^{-2}(1 + x/2)^{-2} = \frac{1}{4}[1 - 2(x/2) + 3(x/2)^2 - 4(x/2)^3]$ M1
2. $= \mathbf{\frac{1}{4} - x/4 + 3x^2/16 - x^3/8}$ A1
3. **valid for $|x| < 2$** B1

WORKED EXAMPLE

Express $f(x) = (5x + 1)/[(x - 1)(x + 2)]$ in partial fractions and hence expand f(x) up to x^2.

1. $f(x) = 2/(x - 1) + 3/(x + 2)$ B1
2. $2/(x - 1) = -2(1 - x)^{-1} = -2(1 + x + x^2 + \dots)$; $3/(x + 2) = (3/2)(1 + x/2)^{-1} = 3/2 - 3x/4 + 3x^2/8$ M1 A1
3. **$f(x) = -\frac{1}{2} - 11x/4 - 13x^2/8 + \dots$, valid for $|x| < 1$ (the stricter condition)** A1

COMMON MISTAKE

Expanding $(2 + x)^{-2}$ as $2^{-2}(1 + x)^{-2}$, forgetting to divide x by 2 inside. Or using x instead of 2x, as in the 2024 Paper 3 report.

LINKING QUESTION

Partial fractions are used to integrate in Further integration and to solve differential equations; the factor theorem gives complex roots in Complex numbers; and the positive-integer binomial theorem is in Series.

Test yourself — cover the answers first

QUICK CHECK

1. Solve $|x - 3| < 5$.
2. What is the remainder when $x^3 - 2x + 4$ is divided by $(x + 1)$?
3. What form do you use for $1/[(x + 2)(x^2 + 3)]$?
4. For which x is the expansion of $(1 + 3x)^{1/3}$ valid?

5. Find the first two terms of $(1 + x)^{-1}$.

EXAM-STYLE

6. Solve the inequality $|x + 2| > |2x - 1|$.
7. The polynomial $2x^3 + ax^2 + bx + 3$ has a factor $(2x - 1)$ and leaves remainder 5 when divided by $(x - 1)$. Find a and b.
8. Expand $(1 + 2x)/(1 - x)^2$ in ascending powers of x up to x^2.

Show answers

1. $-2 < x < 8$
2. $p(-1) = 5$
3. $A/(x + 2) + (Bx + C)/(x^2 + 3)$
4. $|x| < \frac{1}{3}$
5. $1 - x$
6. $(x + 2)^2 > (2x - 1)^2 \Rightarrow 3x^2 - 8x - 3 < 0 \Rightarrow (3x + 1)(x - 3) < 0 \Rightarrow -\frac{1}{3} < x < 3$
7. $p(\frac{1}{2}) = 0 \Rightarrow \frac{1}{4} + a/4 + b/2 + 3 = 0 \Rightarrow a + 2b = -13$; $p(1) = 5 \Rightarrow a + b = 0 \Rightarrow b = -13, a = 13$
8. $(1 + 2x)(1 + 2x + 3x^2 + \ldots) = 1 + 4x + 7x^2 + \ldots$

Stuck on a question?

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