A LEVEL MATHS 9709 · PAPER 3 PURE MATHEMATICS 3 · 3.9 · ~6 MIN READ

Complex numbers

Arithmetic with complex numbers, conjugates, modulus and argument, the Argand diagram, square roots of a complex number, conjugate roots of real polynomials, the forms $r(\cos\theta + i\sin\theta)$ and $re^{i\theta}$, and loci and regions.

i and x + iy | Conjugate | Division | Modulus & argument | Argand diagram | Square roots | Conjugate-pair roots | Polar & exponential form

RS by Riyaz Sir
mentorriyaz.in

Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded

highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **$i^2 = -1$.** Treat x + iy like an algebraic expression, replacing i^2 by −1.
- **To divide, multiply top and bottom by the conjugate** of the bottom; the denominator becomes real.
- **$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$:** multiplying multiplies the moduli and adds the arguments; dividing divides the moduli and subtracts the arguments.
- **Loci:** $|z - a| = k$ is a circle; $|z - a| = |z - b|$ is a perpendicular bisector; $\arg(z - a) = \alpha$ is a half-line from a.

BY THE END YOU CAN

- Add, subtract, multiply and divide complex numbers in the form x + iy, and use conjugates.
- Find the modulus and argument, using $-\pi < \theta \leq \pi$, and represent numbers on an Argand diagram.
- Find square roots of a complex number and solve polynomial equations with complex roots.
- Use the forms $r(\cos\theta + i\sin\theta)$ and $re^{i\theta}$ to multiply and divide.
- Sketch loci and regions defined by modulus and argument.

Riyaz Sir

1 Cartesian form

Arithmetic

Conjugate — of $z = x + iy$ is $z^* = x - iy$. $zz^* = x^2 + y^2$ is real.

- **Equal complex numbers:** real parts equal and imaginary parts equal. This gives two equations from one.

WORKED EXAMPLE

Express $(3 + 4i)/(1 - 2i)$ in the form $x + iy$.

1. multiply by $(1 + 2i)/(1 + 2i)$ M1
2. $(3 + 6i + 4i - 8)/(1 + 4) = (-5 + 10i)/5$ A1
3. **$-1 + 2i$** A1

WORKED EXAMPLE

Find z such that $2z + iz^* = 3 + 6i$.

1. $z = x + iy$: $2x + 2iy + ix + y = 3 + 6i$ M1
2. real: $2x + y = 3$; imaginary: $x + 2y = 6$ M1
3. **$x = 0$, $y = 3$: $z = 3i$** A1

WORKED EXAMPLE

Find the square roots of $5 - 12i$.

1. $(a + ib)^2 = 5 - 12i \Rightarrow a^2 - b^2 = 5$, $2ab = -12$ M1
2. $b = -6/a \Rightarrow a^4 - 5a^2 - 36 = 0 \Rightarrow a^2 = 9$ M1
3. **$\pm(3 - 2i)$** A1 A1

Polynomial roots

- **Real coefficients ⇒ non-real roots come in conjugate pairs.** If $1 + 2i$ is a root, so is $1 - 2i$, and $(z - 1)^2 + 4 = z^2 - 2z + 5$ is a factor.

WORKED EXAMPLE

Given that $1 + 2i$ is a root of $z^3 - 4z^2 + 9z - 10 = 0$, find the other roots.

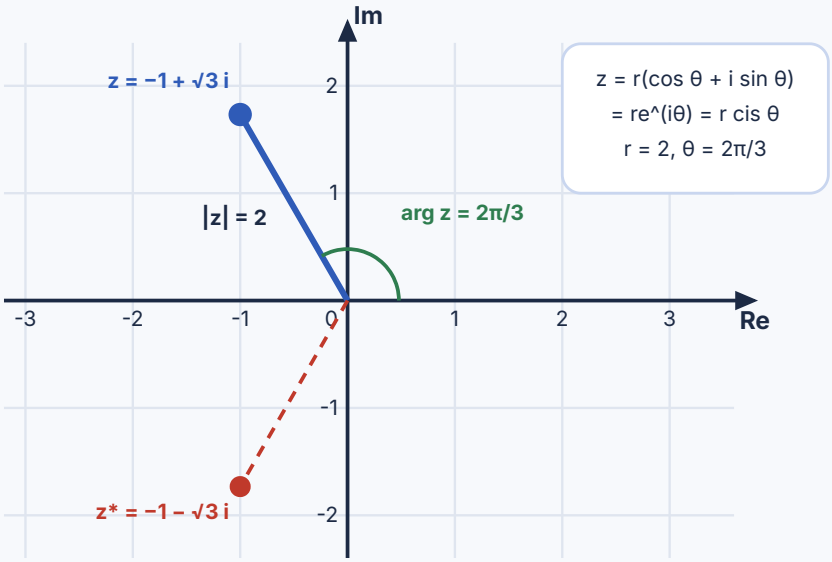
1. $1 - 2i$ is also a root B1
2. quadratic factor $z^2 - 2z + 5$ M1 A1
3. **$z^3 - 4z^2 + 9z - 10 = (z^2 - 2z + 5)(z - 2)$, so the roots are $2, 1 \pm 2i$** A1

COMMON MISTAKE

Calculator answers without working. You may check on a calculator, but division and roots must be shown algebraically.

2 Modulus and argument

The Argand diagram



z = x + iy is the point (x, y); r is its distance from O and θ its angle from the positive real axis.

$|z| = \sqrt{(x^2 + y^2)}$; arg z = θ, with tan θ = y/x and $-\pi < \theta \leq \pi$

always sketch to choose the quadrant

- **Quadrants:** for −1 + √3 i (second quadrant), $\tan^{-1}(\sqrt{3}/1) = \pi/3$, so arg z = π − π/3 = **2π/3**, and |z| = 2.

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

polar and exponential forms

$|z_1z_2| = |z_1||z_2|$, $\arg(z_1z_2) = \arg z_1 + \arg z_2$; $|z_1/z_2| = |z_1|/|z_2|$, $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ LEARN IT

adjust by 2π to stay in (−π, π]

WORKED EXAMPLE

$z = 2e^{i\pi/3}$ and $w = \sqrt{2}\, e^{-i\pi/4}$. Find zw and z/w in the form $re^{i\theta}$.

1. $zw = 2\sqrt{2}\, e^{i(\pi/3 - \pi/4)} = \mathbf{2\sqrt{2}\, e^{i\pi/12}}$ M1 A1
2. **$z/w = \sqrt{2}\, e^{i(\pi/3 + \pi/4)} = \sqrt{2}\, e^{7i\pi/12}$** A1

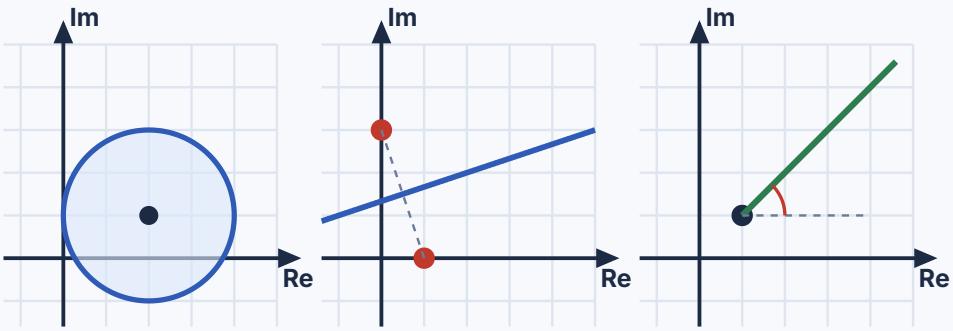
COMMON MISTAKE

Arguments from $\tan^{-1}(y/x)$ alone. For −1 − i, $\tan^{-1}(1) = \pi/4$ but the argument is −3π/4. The 2024 Paper 3 report flagged quadrant errors and degree answers.

3 Loci and regions

Three standard loci

Three standard loci on an Argand diagram



|z – (2 + i)| ≤ 2
disc, centre 2 + i, radius 2

|z – 1| = |z – 3i|
perpendicular bisector

arg(z – 1 – i) = π/4
half-line from 1 + i

Fixed distance · equal distances · fixed angle

A circle (fixed distance), a perpendicular bisector (equal distances) and a half-line (fixed angle).

- **|z – (2 + i)| ≤ 2:** all points within 2 of 2 + i: a disc. Strict "<" uses a dashed boundary.
- **|z – 1| = |z – 3i|:** points equidistant from 1 and 3i: the perpendicular bisector of the segment joining them.
- **arg(z – 1 – i) = π/4:** a half-line starting at 1 + i (not including that point) at 45° to the horizontal.

WORKED EXAMPLE

Find the Cartesian equation of |z – 1| = |z – 3i|.

1. $(x - 1)^2 + y^2 = x^2 + (y - 3)^2$ M1
2. $-2x + 1 = -6y + 9$ A1
3. **$x - 3y + 4 = 0$ ($y = (x + 4)/3$)** A1

WORKED EXAMPLE

z lies on the circle |z – 3 – 4i| = 2. Find the least and greatest values of |z|.

1. the centre 3 + 4i is 5 from O M1
2. **least = 5 − 2 = 3, greatest = 5 + 2 = 7** A1 A1

COMMON MISTAKE

Drawing arg(z – a) = α as a full line. It is a half-line from a, and a is excluded.

LINKING QUESTION

Modulus and argument link to 2D vectors; the factor theorem used here is in Algebra; and the trig values come from Further trigonometry.

Test yourself — cover the answers first

QUICK CHECK

1. Simplify i^3.
2. Find $(2 + i)(2 - i)$.
3. Find $|-5 + 12i|$.
4. Find $\arg(-2)$, with $-\pi < \theta \leq \pi$.
5. If $3 - i$ is a root of a real cubic, name another root.

EXAM-STYLE

6. Express $(1 + i)^4$ in the form $x + iy$ using the exponential form.
7. Find the modulus and argument of $(1 + i\sqrt{3})/(1 - i)$.
8. Shade the region $|z - 2| \leq 2$ and $\text{Re}\, z \geq 1$: describe it.

Show answers

1. $-i$
2. 5
3. 13
4. π
5. $3 + i$
6. $(\sqrt{2}\, e^{i\pi/4})^4 = 4e^{i\pi} = -4$
7. modulus $2/\sqrt{2} = \sqrt{2}$; argument $\pi/3 + \pi/4 = 7\pi/12$
8. inside the disc centre 2, radius 2, to the right of the line $x = 1$

Stuck on a question?

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