A LEVEL MATHS 9709 · PAPER 3 PURE MATHEMATICS 3 · 3.8 · ~5 MIN READ

Differential equations

Forming a differential equation from a rate-of-change statement, solving first-order equations by separating the variables, using a condition to find the particular solution, and interpreting the result in context.

Forming equations | Proportionality | Separating variables | General solution | Particular solution | Exponential growth & decay | Partial fractions in DEs | Interpreting solutions

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Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded

highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **A differential equation links a rate to a quantity.** "The rate of decrease of N is proportional to N" means dN/dt = −kN with k > 0.
- **Separate the variables:** get all y-terms with dy on one side and all x-terms with dx on the other, then integrate both sides.
- **One constant is enough.** Add + c on one side only, then use the given condition straight away.
- **Interpret:** check the long-term behaviour (what happens as t → ∞) and answer in the context's units.

BY THE END YOU CAN

- Form a simple differential equation from a statement about rates of change.
- Solve first-order equations with separable variables, using any integration technique from 3.5.
- Find a particular solution from given conditions.
- Interpret the solution and its behaviour in context.

1 Forming the equation

Words into symbols

Statement	Equation
The rate of increase of P is proportional to P	$\frac{dP}{dt} = kP$
The temperature θ falls at a rate proportional to its excess over 20 °C	$\frac{d\theta}{dt} = -k(\theta - 20)$
The volume decreases at a rate proportional to √V	$\frac{dV}{dt} = -k\sqrt{V}$
The gradient of the curve at (x, y) is xy^2	$\frac{dy}{dx} = xy^2$

- **Constants:** state that k > 0 when the quantity decreases, and put the minus sign in the equation.

COMMON MISTAKE

Missing the minus sign for a decreasing quantity, or writing dV/dx when the variable is time.

2 Separating the variables

Method

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

then add one constant

WORKED EXAMPLE

Solve $dy/dx = 2xy$, given $y = 3$ when $x = 0$.

1. $\int (1/y)\,dy = \int 2x\,dx$ M1
2. $\ln y = x^2 + c$ A1
3. $x = 0, y = 3 \Rightarrow c = \ln 3$ M1
4. **$y = e^{x^2 + \ln 3} = 3e^{x^2}$** A1

WORKED EXAMPLE

Solve $x\,dy/dx = y^2 + 1$ for $x > 0$, given $y = 1$ when $x = 1$. Give y in terms of x.

1. $\int 1/(y^2 + 1)\,dy = \int 1/x\,dx$ M1
2. $\tan^{-1} y = \ln x + c$ A1 A1
3. $x = 1, y = 1$: $\pi/4 = 0 + c$ M1
4. **$y = \tan(\ln x + \pi/4)$** A1

WORKED EXAMPLE

Solve $(x + 1)(x + 2)\,dy/dx = y$ for $x \geq 0$, given $y = 2$ when $x = 0$.

1. $\int (1/y)\,dy = \int 1/[(x + 1)(x + 2)]\,dx = \int [1/(x + 1) - 1/(x + 2)]\,dx$ M1 A1
2. $\ln y = \ln(x + 1) - \ln(x + 2) + c$ A1
3. $\ln 2 = \ln ½ + c \Rightarrow c = \ln 4$ M1
4. **$y = 4(x + 1)/(x + 2)$** A1

- **Removing logs:** write the constant as $\ln A$ so you can combine: $\ln y = \ln[A(x + 1)/(x + 2)]$. It's neater than e^c.

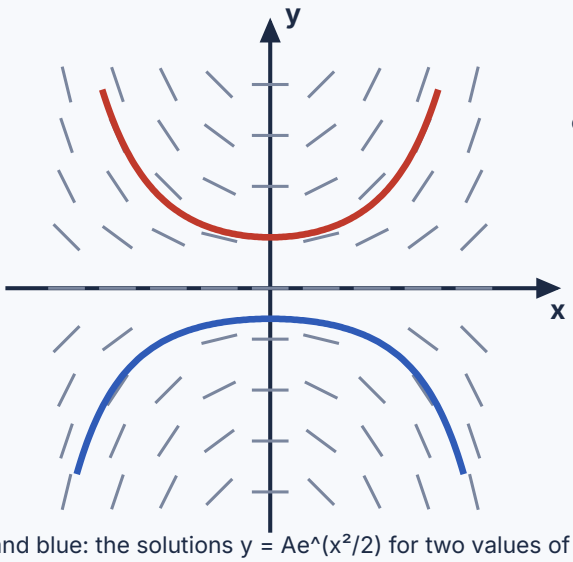
COMMON MISTAKE

Writing $e^{x^2 + c} = e^{x^2} + e^c$. It is $e^{x^2} \times e^c$. The 2024 Paper 3 report highlighted this exponential error.

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3 In context

Growth, decay and cooling



dy/dx = xy

each dash has gradient xy

Red and blue: the solutions y = Ae^(x²/2) for two values of A.

Every solution curve follows the slope field; the initial condition picks one curve.

WORKED EXAMPLE

A liquid cools so that dθ/dt = −k(θ − 20), where θ °C is the temperature after t minutes. θ = 90 when t = 0 and θ = 60 when t = 10. Find how long it takes to reach 30 °C.

1. $\int 1/(\theta - 20)\, d\theta = -\int k\, dt \Rightarrow \ln(\theta - 20) = -kt + c$ M1 A1
2. $t = 0$: $c = \ln 70$, so $\theta = 20 + 70e^{-kt}$ A1
3. $t = 10$: $40 = 70e^{-10k} \Rightarrow k = \ln(7/4)/10 = 0.0560$ M1 A1
4. **$10 = 70e^{-kt} \Rightarrow t = \ln 7/k = 34.8$ minutes** A1

- **Long term:** as $t \to \infty$, $e^{-kt} \to 0$ and $\theta \to 20$ °C, the room temperature. Questions often ask for this "what happens eventually".

COMMON MISTAKE

Rounding k too early. Keep k exact (ln(7/4)/10) or in the calculator memory; rounding to 0.06 changes the final answer to 32.4.

LINKING QUESTION

The integrations here are from Further integration, the partial fractions from Algebra, and the exponential and log rules from Logarithms and exponentials.

Test yourself — cover the answers first

QUICK CHECK

1. Write "the rate of increase of x is proportional to x^3" as an equation.
2. Separate: dy/dx = y cos x.

3. How many arbitrary constants does a first-order DE solution need?

4. Simplify $e^{\ln 5 + 2x}$.

5. In $d\theta/dt = -k(\theta - 15)$, what does θ approach in the long term?

EXAM-STYLE

6. Solve $dy/dx = e^{x - y}$, given $y = 0$ when $x = 0$.

7. Solve $dy/dx = (y + 1)/x$, given $y = 2$ when $x = 1$.

8. A population P satisfies $dP/dt = 0.02P$ with $P = 500$ at $t = 0$. Find t when $P = 1000$.

Show answers

1. $dx/dt = kx^3$

2. $\int(1/y)\, dy = \int \cos x\, dx$

3. one

4. $5e^{2x}$

5. 15

6. $\int e^y\, dy = \int e^x\, dx \Rightarrow e^y = e^x + c$, $c = 0 \Rightarrow y = x$

7. $\ln(y + 1) = \ln x + \ln 3 \Rightarrow y = 3x - 1$

8. $P = 500e^{0.02t}$; $e^{0.02t} = 2 \Rightarrow t = 50 \ln 2 = 34.7$

Stuck on a question?

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