A LEVEL MATHS 9709 · PAPER 1 PURE MATHEMATICS 1 · 1.7 · ~9 MIN READ

Differentiation

The gradient as the limit of chord gradients, differentiating x^n for any rational n, the chain rule, tangents and normals, increasing and decreasing functions, stationary points and connected rates of change.

Gradient as a limit | Power rule | Chain rule | Tangents & normals | Increasing & decreasing | Stationary points | Second derivative | Optimisation

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Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **The derivative dy/dx** is the gradient of the curve at a point: the limit of the gradients of chords as they shrink towards that point.
- **Power rule:** $d/dx(x^n) = nx^{n-1}$ for any rational n. Rewrite roots and fractions as powers first.
- **Chain rule:** $d/dx[(ax + b)^n] = an(ax + b)^{n-1}$; in general dy/dx = dy/du × du/dx.
- **Stationary points** are where dy/dx = 0. If $d^2y/dx^2 < 0$ it is a maximum; if > 0, a minimum.
- **Connected rates:** dy/dt = dy/dx × dx/dt. Write the chain down before substituting numbers.

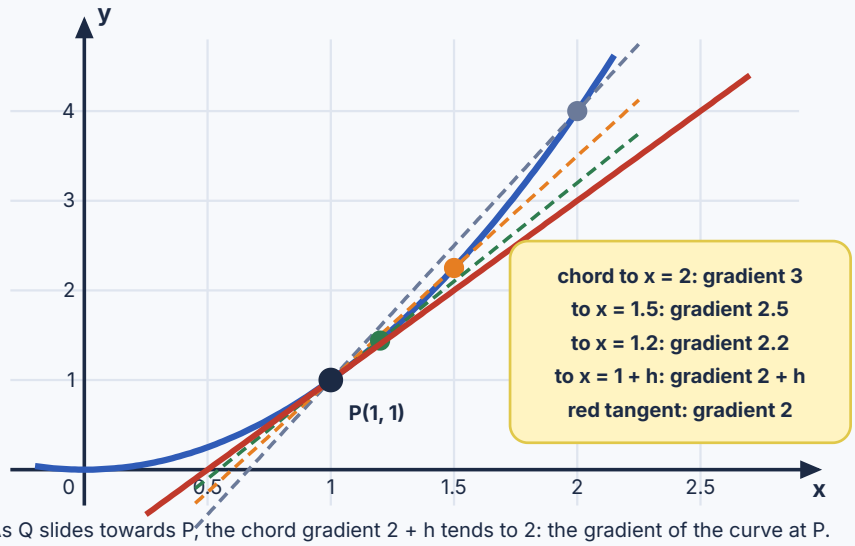
BY THE END YOU CAN

- Explain the gradient of a curve as the limit of chord gradients.
- Differentiate sums of powers of x and composite functions with the chain rule.
- Find equations of tangents and normals.
- Decide where a function is increasing or decreasing, and prove a function is always increasing.
- Find stationary points, determine their nature, and solve optimisation problems.
- Solve connected rates of change problems.

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1 What a derivative is

Chords become a tangent



As Q slides towards P, the chord gradient 2 + h tends to 2: the gradient of the curve at P.

The chord PQ on $y = x^2$ from $x = 1$ to $x = 1 + h$ has gradient $2 + h$. As $h \to 0$ this tends to 2.

- **Gradient of the chord** from (1, 1) to $(1 + h, (1 + h)^2)$: $[(1 + h)^2 - 1]/h = (2h + h^2)/h = 2 + h$. As $h \to 0$, it tends to 2.
- **Notation:** dy/dx or f'(x) is the gradient function; d^2y/dx^2 or f''(x) is the derivative of the derivative (the rate of change of the gradient).
- **Paper 1 only needs this informal idea of a limit.** You won't be asked to differentiate from first principles.

2 Differentiating powers

The power rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$ FORMULA LIST (MF19)

for any rational n; d/dx(k) = 0 for a constant k

- **Constant multiples and sums:** $d/dx[af(x) + bg(x)] = af'(x) + bg'(x)$.
- **Rewrite first:** $\sqrt{x} = x^{1/2}$, $1/x^3 = x^{-3}$, $5/(2x) = (5/2)x^{-1}$. The 2024 report saw 1/(2x) differentiated as $2x^{-1}$: it is $\frac{1}{2}x^{-1}$, with derivative $-\frac{1}{2}x^{-2}$.

WORKED EXAMPLE

Find dy/dx for $y = 4\sqrt{x} - 3/x^2 + 5$.

1. $y = 4x^{1/2} - 3x^{-2} + 5$ M1
2. **$dy/dx = 2x^{-1/2} + 6x^{-3} = 2/\sqrt{x} + 6/x^3$** A1 A1

The chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

for $y = f(u)$ where $u = g(x)$

$$\frac{d}{dx}[(ax + b)^n] = an(ax + b)^{n-1}$$

the most common case in Paper 1

- **Why:** if u changes by a small δu and y by δy, then δy/δx = (δy/δu)(δu/δx). Letting δx → 0 gives the chain rule.

WORKED EXAMPLE

Differentiate $y = (3x - 1)^5$ and $y = 6/\sqrt{(2x + 5)}$.

1. $dy/dx = 5(3x - 1)^4 \times 3 = \mathbf{15(3x - 1)^4}$ B1
2. $y = 6(2x + 5)^{-1/2} \Rightarrow dy/dx = 6 \times (-½)(2x + 5)^{-3/2} \times 2$ M1
3. $= \mathbf{-6(2x + 5)^{-3/2}}$ A1

COMMON MISTAKE

Forgetting the inner derivative (the × 3 or × 2). It is the most common chain-rule error.

3 Tangents and normals

Method

- **1** · Find the y-coordinate at the given x (if not given). **2** · Find dy/dx at that x: this is the tangent gradient m. **3** · The normal gradient is −1/m. **4** · Use $y - y_1 = m(x - x_1)$.

WORKED EXAMPLE

Find the equations of the tangent and the normal to $y = x^3 - 3x$ at the point where x = 2.

1. $y = 8 - 6 = 2$; $dy/dx = 3x^2 - 3 = 9$ at $x = 2$ M1 A1
2. tangent: $y - 2 = 9(x - 2) \Rightarrow \mathbf{y = 9x - 16}$ A1
3. **normal gradient −1/9: $y - 2 = -(1/9)(x - 2) \Rightarrow x + 9y = 20$** M1 A1

4 Increasing and decreasing functions

- **Increasing** where f'(x) > 0; **decreasing** where f'(x) < 0.
- **"Show that f is increasing for all x"** needs an argument for every x, such as completing the square on f'(x). Substituting a few values is not a proof; the 2024 report penalised exactly that.

WORKED EXAMPLE

Find the set of values of x for which $f(x) = x^3 - 6x^2 + 9x + 1$ is decreasing.

1. $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$ M1
2. **decreasing ⇒ $f'(x) < 0$ ⇒ $1 < x < 3$** A1

WORKED EXAMPLE

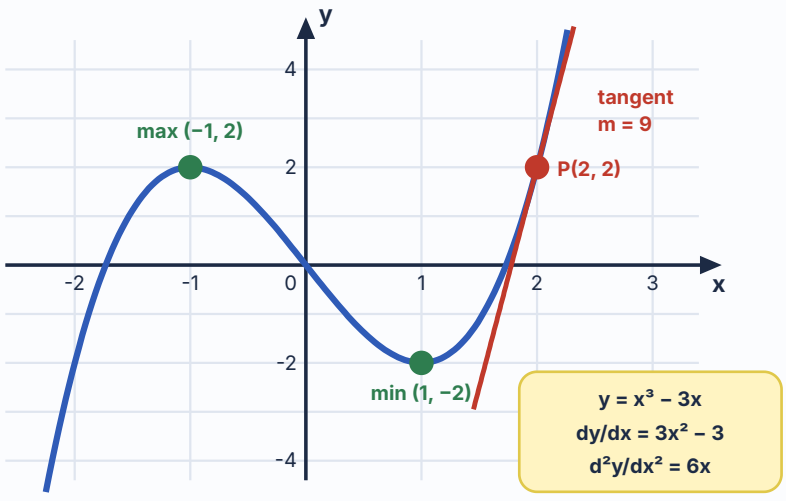
Show that $f(x) = x^3 + 3x^2 + 5x$ is an increasing function.

1. $f'(x) = 3x^2 + 6x + 5$ B1
2. $= 3(x + 1)^2 + 2$ M1
3. **$(x + 1)^2 \geq 0$, so $f'(x) \geq 2 > 0$ for all x, hence f is increasing** A1

5 Stationary points

Finding and classifying

- **Stationary point:** dy/dx = 0. Solve for x, then find y.
- **Second derivative test:** $d^2y/dx^2 < 0$ ⇒ maximum; $d^2y/dx^2 > 0$ ⇒ minimum. If $d^2y/dx^2 = 0$, the test fails: check the sign of dy/dx on either side instead.
- **Points of inflexion** are not in the 9709 syllabus, so you won't be asked to find them.



Stationary points where dy/dx = 0. Sign of d^2y/dx^2 decides: negative → max, positive → min.

$y = x^3 - 3x$: a maximum at (−1, 2), a minimum at (1, −2), and the tangent at (2, 2).

WORKED EXAMPLE

Find the stationary points of $y = x^3 - 3x$ and determine their nature.

1. $dy/dx = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$ M1 A1
2. points $(1, -2)$ and $(-1, 2)$ A1
3. **$d^2y/dx^2 = 6x$: at $x = 1$ it is $6 > 0 \Rightarrow$ minimum $(1, -2)$; at $x = -1$ it is $-6 < 0 \Rightarrow$ maximum $(-1, 2)$** M1 A1

COMMON MISTAKE

Not showing the sign of the second derivative. Write "$d^2y/dx^2 = 6 > 0$, so minimum". Just stating "minimum" can lose the mark.

Optimisation

- **1** · Write the quantity to maximise or minimise in terms of **one** variable, using any constraint given. **2** · Differentiate and set to zero. **3** · Check it is a max or min. **4** · Answer the question asked (often the value of the quantity, not x).

WORKED EXAMPLE

A farmer has 60 m of fencing to make a rectangular pen against a long straight wall, so only three sides need fencing. Find the maximum area.

1. let the sides perpendicular to the wall be y, the parallel side x: $x + 2y = 60$ B1
2. $A = xy = (60 - 2y)y = 60y - 2y^2$ M1
3. $dA/dy = 60 - 4y = 0 \Rightarrow y = 15$, $x = 30$ M1 A1
4. **$d^2A/dy^2 = -4 < 0 \Rightarrow$ maximum; $A = 450\ m^2$** A1

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6 Rates of change

Connected rates

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

chain rule with time

- **Method:** write the formula linking the two quantities, differentiate it, then multiply by the given rate. Label each rate with its meaning and units.

WORKED EXAMPLE

The radius of a spherical balloon increases at 0.2 cm s^{-1}. Find the rate at which its volume increases when r = 5 cm.

1. $V = (4/3)\pi r^3 \Rightarrow dV/dr = 4\pi r^2 = 100\pi$ when $r = 5$ M1
2. $dV/dt = dV/dr \times dr/dt = 100\pi \times 0.2$ M1
3. **$= 20\pi \approx 62.8$ cm^3 s^{-1}** A1

WORKED EXAMPLE

A point moves along $y = \sqrt{(x^2 + 9)}$ so that its x-coordinate increases at 3 units per second. Find the rate of increase of y when x = 4.

1. $dy/dx = x(x^2 + 9)^{-1/2} = 4/5$ at $x = 4$ M1 A1
2. **$dy/dt = (4/5) \times 3 = 2.4$ units per second** A1

LINKING QUESTION

Paper 3 adds e^x, ln x, trig functions, products, quotients and implicit equations in Further differentiation; integration reverses differentiation in Integration; and Mechanics uses derivatives for velocity and acceleration in Kinematics in a straight line.

Test yourself — cover the answers first

QUICK CHECK

1. Differentiate $y = 3x^4 - 2x + 7$.
2. Differentiate $y = 1/\sqrt{x}$.
3. Differentiate $y = (5 - 2x)^3$.
4. What is the gradient of the normal if the tangent gradient is −4?
5. At a stationary point $d^2y/dx^2 = -3$. Is it a max or a min?

EXAM-STYLE

6. The curve $y = 2x + 8/x$ has two stationary points. Find them and determine their nature.
7. A curve has equation $y = (2x - 3)^3 - 6x$. (a) Find dy/dx and d^2y/dx^2. (b) Find the x-coordinates of the stationary points and their nature.
8. The volume of water in a tank is $V = 20h^2 + 30h$ cm^3 when the depth is h cm. Water is added at 70 cm^3 s^{-1}. Find the rate at which h is increasing when h = 4.

Show answers

1. $12x^3 - 2$
2. $-\frac{1}{2}x^{-3/2}$
3. $-6(5 - 2x)^2$
4. $\frac{1}{4}$
5. maximum
6. $dy/dx = 2 - 8/x^2 = 0 \Rightarrow x = \pm 2$; (2, 8) minimum ($d^2y/dx^2 = 16/x^3 = 2 > 0$); (−2, −8) maximum
7. (a) $6(2x - 3)^2 - 6$; $24(2x - 3)$
   (b) $(2x - 3)^2 = 1 \Rightarrow x = 1$ (max, $d^2y/dx^2 = -24$) and $x = 2$ (min, +24)
8. $dV/dh = 40h + 30 = 190$; $dh/dt = 70/190 = 0.368$ cm s^{-1}

Stuck on a question?

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