A LEVEL MATHS 9709 · PAPER 1 PURE MATHEMATICS 1 · 1.2 · ~11 MIN READ

Functions and transformations

Domain and range, one-one functions, inverse functions and their graphs, composite functions and when they exist, and the four graph transformations described in the exact words examiners credit.

Domain & range | One-one functions | Inverse functions | Composite functions | Translations | Stretches | Reflections | Combined transformations

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Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded

highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **A function** sends every input in its **domain** to exactly one output. The set of outputs is the **range**.
- **Only a one-one function has an inverse.** The graph of f^{-1} is the reflection of the graph of f in y = x, and the domain and range swap over.
- **fg(x) means f(g(x)): do g first.** It only exists when every output of g is an allowed input for f.
- **Transformations:** changes outside f act on y in the obvious way; changes inside f act on x in the opposite way. Always name the type ("translation", "stretch", "reflection") with its vector, factor or line.

BY THE END YOU CAN

- Find the range of a function, including quadratics on a restricted domain.
- Decide whether a function is one-one and find its inverse, with the correct domain.
- Form composite functions, decide whether they exist, and find their ranges.
- Describe and apply translations, stretches and reflections, including combinations.

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1 Domain, range and one-one functions

Key words

Domain — the set of input values (x-values) the function is defined for. It is part of the definition: $f(x) = x^2$ for $x \geq 0$ is a different function from $f(x) = x^2$ for $x \in \mathbb{R}$.

Range — the set of output values (y-values) the function actually produces.

One-one function — every output comes from exactly one input. Test: every horizontal line meets the graph at most once.

Many-one function — some output comes from two or more inputs, e.g. $f(x) = x^2$ on $\mathbb{R}$, since $f(2) = f(-2) = 4$.

- **Finding a range:** sketch the graph over the given domain. For quadratics, complete the square to find the vertex, then check the endpoints of the domain.
- **Notation:** write the range in terms of f(x) or y, e.g. $f(x) \geq 2$, not $x \geq 2$. The June 2024 report flagged incomplete range inequalities.

WORKED EXAMPLE

$f(x) = x^2 - 6x + 11$ for $x \geq 3$. Find the range of f.

1. $x^2 - 6x + 11 = (x - 3)^2 + 2$ M1
2. the vertex (3, 2) is at the left end of the domain and the graph rises from there
3. **range: $f(x) \geq 2$** A1

WORKED EXAMPLE

$f(x) = x^2 + 4x$ for $x \leq k$. Find the largest value of k for which f is one-one.

1. $x^2 + 4x = (x + 2)^2 - 4$, so the vertex is at $x = -2$ M1
2. f is one-one on one side of the vertex only
3. **largest k: $k = -2$** A1

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2 Inverse functions

Finding f^{-1}

- **1** · Write y = f(x). **2** · Rearrange to make x the subject. **3** · Swap x and y (or replace y by x) to get $f^{-1}(x)$. **4** · State the domain of f^{-1}: it is the **range of f**.

domain of f^{-1} = range of f, range of f^{-1} = domain of f **LEARN IT**

and $f^{-1}f(x) = ff^{-1}(x) = x$

WORKED EXAMPLE

$f(x) = x^2 - 6x + 11$ for $x \geq 3$. Find $f^{-1}(x)$ and state its domain.

1. $y = (x - 3)^2 + 2 \Rightarrow (x - 3)^2 = y - 2$ **M1**
2. $x - 3 = +\sqrt{(y - 2)}$: take the **positive** root because $x \geq 3$ **M1**
3. **$f^{-1}(x) = 3 + \sqrt{(x - 2)}$, for $x \geq 2$** **A1** **B1**
4. **check: $f(4) = 3$ and $f^{-1}(3) = 3 + 1 = 4$ ✓**

WORKED EXAMPLE

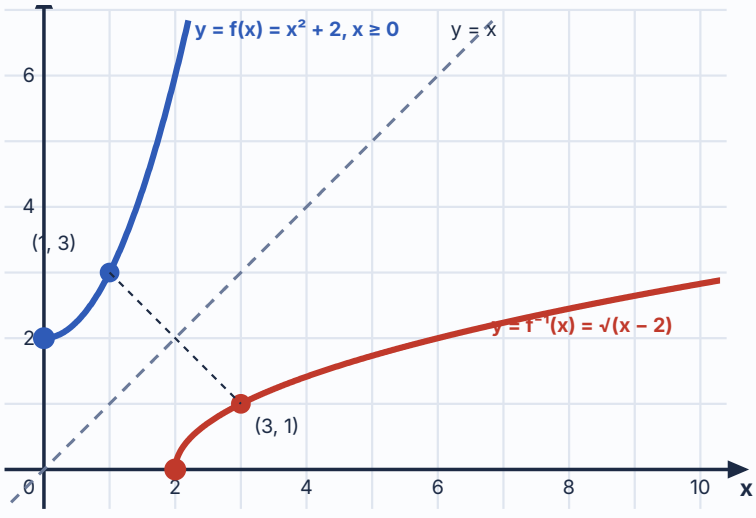
$g(x) = (2x + 3)/(x - 1)$ for $x > 1$. Find $g^{-1}(x)$.

1. $y(x - 1) = 2x + 3 \Rightarrow xy - y = 2x + 3$ **M1**
2. $xy - 2x = y + 3 \Rightarrow x(y - 2) = y + 3 \Rightarrow x = (y + 3)/(y - 2)$ **M1**
3. **$g^{-1}(x) = (x + 3)/(x - 2)$, $x > 2$** **A1** **(the range of g is $g(x) > 2$)**

COMMON MISTAKE

Choosing the wrong sign of the square root. For $x \geq 3$, $f^{-1}(x) = 3 - \sqrt{(x - 2)}$ is wrong: it gives outputs below 3. The 2024 report saw $f^{-1}(x) = -\sqrt{(x + 4)}$-type answers that ignored the range of the inverse.

The graph of f^{-1}



Domain of f^{-1} = range of f ($x \geq 2$). Range of f^{-1} = domain of f ($y \geq 0$). Mirror line y = x.

f and f^{-1} are reflections of each other in y = x: the point (1, 3) on f becomes (3, 1) on f^{-1}.

- **Draw the line y = x** on any sketch of f and f^{-1}, and use the same scale on both axes. Examiners deducted marks when the mirror line was missing.
- **Where do f and f^{-1} meet?** For an increasing function, any intersection lies on y = x, so you can solve f(x) = x instead of f(x) = f^{-1}(x).

EXAM TIP

"Explain why f has no inverse" needs one sentence: "f is not one-one" (for example, f(1) = f(5)). That is the accepted reason.

3 Composite functions

fg(x) = f(g(x))

- **Do the inside function first.** fg(x): apply g, then apply f to the result. gf is usually different.
- **fg exists only if the range of g lies inside the domain of f.**
- **Range of fg:** find the range of g, then find what f does to that set. Don't just simplify the formula and guess.

WORKED EXAMPLE

f(x) = 2x + 1 and g(x) = x^2 − 3, both for x ∈ ℝ. Find fg(x) and gf(x), and solve gf(x) = 6.

1. fg(x) = f(x^2 − 3) = 2(x^2 − 3) + 1 = **$2x^2$ − 5** B1
2. gf(x) = g(2x + 1) = **$(2x + 1)^2$ − 3** B1
3. $(2x + 1)^2$ − 3 = 6 ⇒ $(2x + 1)^2$ = 9 ⇒ 2x + 1 = ±3 M1
4. **x = 1 or x = −2** A1

WORKED EXAMPLE

f(x) = 1/(x − 2) for x > 2 and g(x) = x^2 + 1 for x ∈ ℝ. Explain why fg does not exist, and find the range of gf.

1. range of g is g(x) ≥ 1, but f needs inputs greater than 2 (e.g. g(0) = 1 is not in the domain of f), so **fg does not exist** B1
2. range of f is f(x) > 0, which is inside the domain of g, so gf exists: gf(x) = $1/(x − 2)^2$ + 1 M1
3. **$1/(x − 2)^2$ takes every positive value, so gf(x) > 1** A1

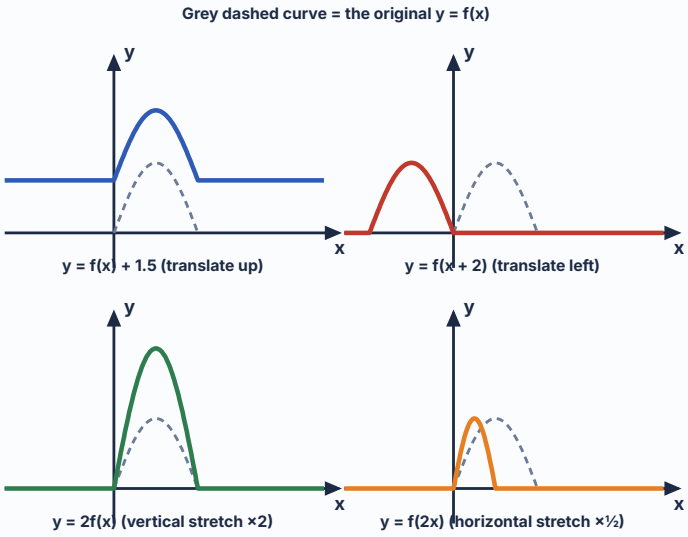
COMMON MISTAKE

Writing fg(x) as f(x) × g(x). The notation means "f of g of x", not a product.

4 Transformations of graphs

The four basic transformations

Equation	Transformation (use these words)	Effect on a point (x, y)
y = f(x) + a	translation by the vector $\begin{pmatrix}0\\a\end{pmatrix}$	(x, y + a)
y = f(x + a)	translation by the vector $\begin{pmatrix}-a\\0\end{pmatrix}$	(x − a, y)
y = af(x)	stretch parallel to the y-axis, scale factor a	(x, ay)
y = f(ax)	stretch parallel to the x-axis, scale factor 1/a	(x/a, y)
y = −f(x)	reflection in the x-axis	(x, −y)
y = f(−x)	reflection in the y-axis	(−x, y)



Outside the brackets: acts on y as written. Inside the brackets: acts on x the opposite way (f(x + 2) moves left; f(2x) squashes by ½).

COMMON MISTAKE

Vague descriptions. "Moves down 2" or "shifts" lose marks. Write "translation by the vector $\begin{pmatrix}0\\-2\end{pmatrix}$". For a stretch, give the direction and the factor: "stretch parallel to the x-axis, scale factor ½".

Combinations: order matters

- **Vertical and horizontal changes are independent,** so their order doesn't matter: y = f(2x) − 3 is a stretch parallel to the x-axis (factor ½) and a translation by $\begin{pmatrix}0\\-3\end{pmatrix}$, in either order.
- **Two changes in the same direction do depend on order.** y = 2f(x) + 3 is "stretch ×2 then translate up 3". y = 2[f(x) + 3] is "translate up 3 then stretch ×2" (the curve ends up 6 higher).
- **Work from the equation outwards:** for y = af(x) + b, deal with the multiplication before the addition, as you would when calculating.

WORKED EXAMPLE

Describe a sequence of transformations that maps $y = x^2$ onto $y = 3(x + 1)^2 - 4$.

1. translation by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ gives $y = (x + 1)^2$ **B1**
2. stretch parallel to the y-axis, scale factor 3, gives $y = 3(x + 1)^2$ **B1**
3. translation by $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ gives $y = 3(x + 1)^2 - 4$ **B1**
4. **(the stretch must come before the downward translation)**

WORKED EXAMPLE

The curve $y = x^2 - 4x$ is reflected in the x-axis and then translated by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Find the equation of the new curve.

1. reflection: $y = -(x^2 - 4x)$ **B1**
2. translation by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$: replace x by $x - 3$ **M1**
3. $y = -[(x - 3)^2 - 4(x - 3)] = -(x^2 - 10x + 21)$
4. **$y = -x^2 + 10x - 21$** **A1**

COMMON MISTAKE

Reversing the order or replacing x by x/2 instead of 2x. The June 2024 report listed both as common errors. Test one point: under $y = f(2x)$, the point (4, 1) on $y = f(x)$ moves to (2, 1).

Trigonometric functions

- **The same rules apply to trig graphs:** $y = 3 - 2 \sin x$ is $y = \sin x$ stretched by factor 2 parallel to the y-axis, reflected in the x-axis, then translated by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.
- **Range of a sin or cos expression:** sin x lies between −1 and 1, so $3 - 2 \sin x$ lies between **1 and 5**.

LINKING QUESTION

The completed square from Quadratics gives ranges and inverses; trig transformations come back in Trigonometry; and inverse functions are why e^x and ln x are reflections of each other in Logarithms and exponential functions.

Test yourself — cover the answers first

QUICK CHECK

1. What condition must a function satisfy to have an inverse?
2. $f(x) = 3x - 2$. Find $f^{-1}(x)$.
3. What is the domain of f^{-1} in terms of f?
4. Describe the transformation $y = f(x) \rightarrow y = f(x - 5)$.
5. $f(x) = x + 4$, $g(x) = 2x$. Find gf(1) and fg(1).

EXAM-STYLE

6. $f(x) = 4 - (x - 2)^2$ for $x \geq 2$. (a) State the range of f. (b) Find $f^{-1}(x)$ and its domain. (c) Sketch $y = f(x)$ and $y = f^{-1}(x)$ on one diagram.
7. Describe fully two transformations that map $y = \cos x$ onto $y = 2 \cos(x/3)$.

8. $f(x) = 1/(x + 1)$ for $x > -1$ and $g(x) = 2x - 1$ for $x > a$. Find the least value of a for which fg exists.

Show answers

1. it must be one-one
2. $(x + 2)/3$
3. the range of f
4. translation by the vector $(5, 0)$
5. $gf(1) = 10$; $fg(1) = 6$
6. (a) $f(x) \leq 4$
   (b) $f^{-1}(x) = 2 + \sqrt{(4 - x)}$, $x \leq 4$
   (c) reflections of each other in $y = x$, meeting on that line
7. stretch parallel to the y-axis, scale factor 2; stretch parallel to the x-axis, scale factor 3
8. need $2x - 1 > -1$ for every input, so $x > 0$: $a = 0$

Stuck on a question?

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