A LEVEL MATHS 9709 · PAPER 3 PURE MATHEMATICS 3 · 3.4 (ALSO PAPER 2) · ~6 MIN READ

Further differentiation

Derivatives of e^x, ln x, sin x, cos x, tan x and $\tan^{-1}x$, the chain, product and quotient rules together, implicit differentiation and parametric differentiation, with tangents, normals and stationary points.

Derivatives of e^x & ln x | Trig derivatives | $\tan^{-1}x$ | Chain rule | Product rule | Quotient rule | Implicit differentiation | Parametric differentiation

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Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded

highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **Six standard derivatives:** $e^x \to e^x$; ln x → 1/x; sin x → cos x; cos x → −sin x; tan x → $\sec^2 x$; $\tan^{-1}x \to 1/(1+x^2)$. Angles are in **radians**.
- **Product rule:** (uv)' = u'v + uv'. **Quotient rule:** (u/v)' = (u'v − uv')/v^2. **Chain rule:** multiply by the derivative of the inside.
- **Implicit differentiation:** differentiate every term with respect to x; each y-term gets a factor dy/dx (chain rule).
- **Parametric:** dy/dx = (dy/dt) ÷ (dx/dt).

BY THE END YOU CAN

- Differentiate exponential, logarithmic and trigonometric functions, including $\tan^{-1}x$, and composites of them.
- Use the product and quotient rules.
- Find dy/dx for curves defined implicitly or parametrically.
- Find tangents, normals and stationary points for these curves.

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1 Standard derivatives

The list

f(x)	f'(x)	With the chain rule
e^x	e^x	$e^{ax+b} \to a\,e^{ax+b}$; $e^{g(x)} \to g'(x)e^{g(x)}$
ln x	1/x	$\ln(ax+b) \to a/(ax+b)$; $\ln g(x) \to g'(x)/g(x)$
sin x	cos x	$\sin(ax+b) \to a\cos(ax+b)$
cos x	−sin x	$\cos(ax+b) \to -a\sin(ax+b)$
tan x	$\sec^2 x$	$\tan(ax) \to a\sec^2(ax)$
$\tan^{-1}x$	$1/(1+x^2)$	$\tan^{-1}(ax) \to a/(1+a^2x^2)$

- **All of these are in the formula list** FORMULA LIST (MF19). The skill is combining them with the rules.
- **Why d/dx $\tan^{-1}x = 1/(1+x^2)$:** if $y = \tan^{-1}x$ then $\tan y = x$. Differentiating implicitly: $\sec^2 y \times dy/dx = 1$, so $dy/dx = 1/\sec^2 y = 1/(1+\tan^2 y) = 1/(1+x^2)$.
- **Logs first:** simplify $\ln(x^3(2x+1)) = 3\ln x + \ln(2x+1)$ before differentiating; it's quicker and safer.

WORKED EXAMPLE

Differentiate (a) e^{3x^2} (b) $\ln(2x+1)$ (c) $\sin^2 x$ (d) $\tan^{-1}(2x)$.

1. (a) **$6x\,e^{3x^2}$** B1
2. (b) **$2/(2x+1)$** B1
3. (c) **$2\sin x \cos x$** (= $\sin 2x$) B1
4. **(d) $2/(1+4x^2)$** B1

COMMON MISTAKE

Using double-angle formulae to differentiate sin 2x. Just use the chain rule: 2 cos 2x. The 2024 Paper 3 report advises against unnecessary identities.

2 Product and quotient rules

The rules

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$ FORMULA LIST (MF19)

product rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ FORMULA LIST (MF19)

quotient rule

WORKED EXAMPLE

Find the x-coordinate of the stationary point of $y = x^2 \ln x$, and its y-coordinate exactly.

1. $dy/dx = 2x \ln x + x^2 \times (1/x) = 2x \ln x + x$ M1 A1
2. $x(2 \ln x + 1) = 0$; $x > 0$, so $\ln x = -\frac{1}{2}$ M1
3. **$x = e^{-1/2}$, $y = e^{-1} \times (-\frac{1}{2}) = -1/(2e)$** A1

WORKED EXAMPLE

Differentiate $y = (x^2 + 1)/(x - 2)$.

1. $dy/dx = [2x(x - 2) - (x^2 + 1)(1)]/(x - 2)^2$ M1 A1
2. **$= (x^2 - 4x - 1)/(x - 2)^2$** A1

WORKED EXAMPLE

Find the x-coordinate of the stationary point of $y = e^{2x} \sin x$ for $0 < x < \pi$.

1. $dy/dx = 2e^{2x} \sin x + e^{2x} \cos x = e^{2x}(2 \sin x + \cos x)$ M1 A1
2. $e^{2x} \neq 0 \Rightarrow \tan x = -\frac{1}{2}$ M1
3. **in $(0, \pi)$: $x = \pi - 0.4636 = 2.68$** A1

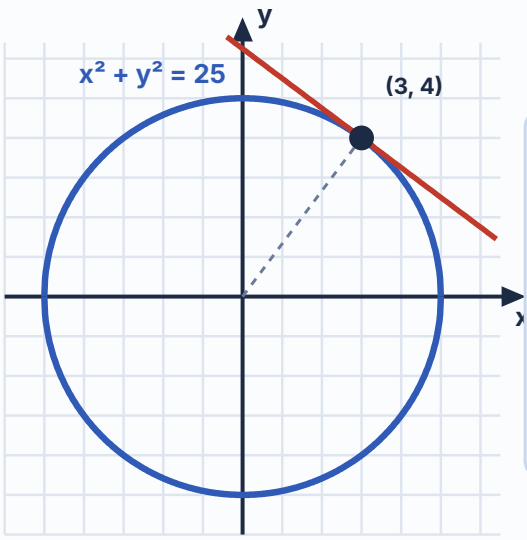
COMMON MISTAKE

Quotient-rule sign errors (v u' − u v', in that order) and missing the chain rule inside, e.g. for $e^{\sin x}$ or $\cos^2 x$. Both were common in 2024.

3 Implicit differentiation

Every term, with dy/dx on the y-terms

- **$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$; $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$** (product rule); $\frac{d}{dx}(e^y) = e^y \frac{dy}{dx}$; $\frac{d}{dx}(\ln y) = (1/y)\frac{dy}{dx}$.
- **Then collect** the dy/dx terms on one side and factorise.



differentiate every term in x:
2x + 2y dy/dx = 0
dy/dx = −x/y
at (3, 4): dy/dx = −3/4
tangent: y − 4 = −¾(x − 3)
check: the radius has
gradient 4/3, and
4/3 × (−3/4) = −1 ✓

Implicit differentiation of a circle gives the tangent gradient at any point.

WORKED EXAMPLE

The curve $x^2 + 3xy + y^2 = 11$ passes through (1, 2). Find the equation of the tangent there.

1. $2x + 3y + 3x\,dy/dx + 2y\,dy/dx = 0$ M1 A1
2. $dy/dx = -(2x + 3y)/(3x + 2y) = -8/7$ at (1, 2) M1
3. **$y - 2 = -(8/7)(x - 1) \Rightarrow 8x + 7y = 22$** A1

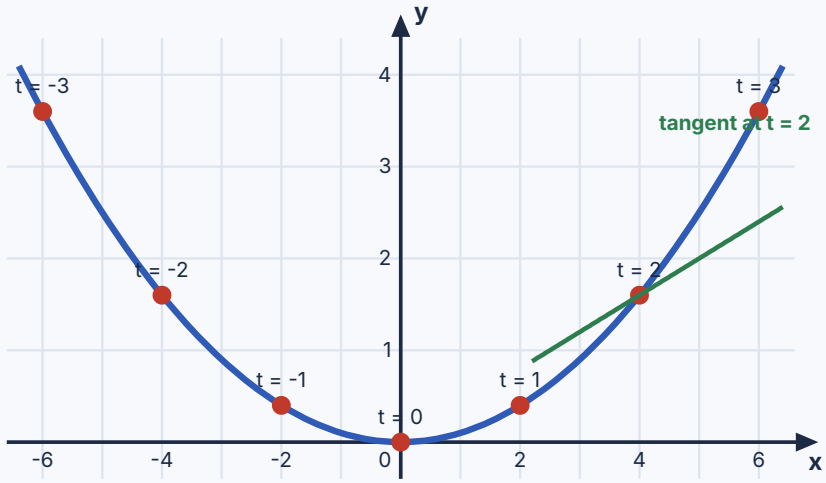
COMMON MISTAKE

Forgetting the product rule on xy or on terms like x ln y. The 2024 Paper 2 and 3 reports both flagged ln-terms not differentiated as products.

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4 Parametric differentiation

dy/dx = (dy/dt)/(dx/dt)



x = 2t, y = t²/2.5: each value of t gives one point (x, y).
dy/dx = (dy/dt) ÷ (dx/dt) = (0.8t) ÷ 2 = 0.4t; at t = 2 the gradient is 0.8.

Each value of t gives a point; the gradient is the ratio of the rates of change.

WORKED EXAMPLE

A curve has parametric equations x = 3 cos t, y = 2 sin t. Find the gradient of the normal at t = π/4.

1. dx/dt = −3 sin t, dy/dt = 2 cos t **B1**
2. dy/dx = 2 cos t/(−3 sin t) = −(2/3) cot t = −2/3 at t = π/4 **M1** **A1**
3. **normal gradient = 3/2** **A1**

- **Finding a point from a condition:** for example, the tangent is horizontal when dy/dt = 0 (and dx/dt ≠ 0); vertical when dx/dt = 0.
- **Converting to Cartesian form** (eliminating t) is sometimes asked: here cos t = x/3 and sin t = y/2, so $x^2/9 + y^2/4 = 1$.

LINKING QUESTION

Paper 1 differentiation is in Differentiation; these derivatives reverse into the integrals of Further integration; and the iteration in Numerical solution of equations often starts from a stationary point equation.

Test yourself — cover the answers first

QUICK CHECK

1. Differentiate e^{-4x}.
2. Differentiate $\ln(x^2 + 5)$.
3. Differentiate tan 3x.
4. Find d/dx(xy) implicitly.
5. $x = t^2$, $y = t^3$. Find dy/dx.

EXAM-STYLE

6. Find the coordinates of the stationary point of $y = x\,e^{-2x}$ and show it is a maximum.
7. The curve $x^3 + y^3 = 9xy$ passes through (2, 4). Find dy/dx at this point.
8. Show that the derivative of $\tan^{-1}(x/2)$ is $2/(4 + x^2)$.

Show answers

1. $-4e^{-4x}$
2. $2x/(x^2 + 5)$
3. $3\sec^2 3x$
4. $y + x\,dy/dx$
5. $3t/2$
6. $dy/dx = e^{-2x}(1 - 2x) = 0 \Rightarrow x = ½$, $y = 1/(2e)$; $d^2y/dx^2 = e^{-2x}(4x - 4) < 0$ at $x = ½$
7. $3x^2 + 3y^2\,dy/dx = 9y + 9x\,dy/dx \Rightarrow dy/dx = (3y - x^2)/(y^2 - 3x) = 8/10 = 0.8$
8. chain rule: $(1/2) \times 1/(1 + x^2/4) = (1/2) \times 4/(4 + x^2) = 2/(4 + x^2)$

Stuck on a question?

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