A LEVEL MATHS 9709 · PAPER 3 PURE MATHEMATICS 3 · 3.5 (ALSO PAPER 2) · ~8 MIN READ

Further integration

Integrating e^x, $1/x$, trig functions and $1/(a^2 + x^2)$, using identities and partial fractions, the $f'(x)/f(x)$ pattern, integration by parts, a given substitution, and the trapezium rule with over- and under-estimates.

Standard integrals | Trig identities | Partial fractions | $f'(x)/f(x)$ | Integration by parts | Substitution | Trapezium rule | Over/under-estimates

RS by Riyaz Sir
mentorriyaz.in

Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded

highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **Reverse the derivative list:** $\int e^{ax+b}\,dx = (1/a)e^{ax+b}$; $\int 1/(ax + b)\,dx = (1/a)\ln|ax + b|$; $\int 1/(a^2 + x^2)\,dx = (1/a)\tan^{-1}(x/a)$.
- **Match the method to the shape:** a product of x and a function → parts; $f'(x)/f(x)$ → $\ln|f(x)|$; a fraction with a factorised denominator → partial fractions; $\sin^2$ or $\cos^2$ → double-angle identity.
- **Substitution:** change every part, x, dx and the limits, into u.
- **Trapezium rule:** $h/2 \times$ [first + last + 2(middle)], with $h = (b - a)/n$. It over-estimates for a curve bending upwards and under-estimates for one bending downwards.

BY THE END YOU CAN

- Integrate e^x, $1/x$, sin, cos, $\sec^2$ and $1/(a^2 + x^2)$ with linear arguments.
- Use trig identities and partial fractions before integrating.
- Recognise and integrate $f'(x)/f(x)$.
- Integrate by parts and by a given substitution, including definite integrals.
- Use the trapezium rule and say whether it gives an over- or under-estimate.

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1 Standard integrals

The list

f(x)	∫f(x) dx
$e^{ax + b}$	$(1/a)\, e^{ax + b} + c$
$1/(ax + b)$	$(1/a) \ln \lvert ax + b \rvert + c$
$\sin(ax + b)$	$-(1/a) \cos(ax + b) + c$
$\cos(ax + b)$	$(1/a) \sin(ax + b) + c$
$\sec^2(ax + b)$	$(1/a) \tan(ax + b) + c$
$1/(a^2 + x^2)$	$(1/a) \tan^{-1}(x/a) + c$

- **These are in the formula list** FORMULA LIST (MF19). The usual error is the factor 1/a: always check by differentiating your answer.
- **The modulus in ln |x|** matters when x can be negative. For definite integrals, the limits must not cross the asymptote.

WORKED EXAMPLE

Find the exact value of $\int_1^3 1/(2x - 1)\, dx$.

1. $[\tfrac{1}{2} \ln(2x - 1)]_1^3$ M1 A1
2. $= \tfrac{1}{2} \ln 5 - \tfrac{1}{2} \ln 1 = \tfrac{1}{2} \ln 5$ A1

WORKED EXAMPLE

Find the exact value of $\int_0^2 1/(4 + x^2)\, dx$.

1. a = 2: $[\tfrac{1}{2} \tan^{-1}(x/2)]_0^2$ M1 A1
2. $= \tfrac{1}{2} \tan^{-1}1 = \tfrac{1}{2} \times \pi/4 = \pi/8$ A1

COMMON MISTAKE

Writing $\int e^{3x}\, dx = 3e^{3x}$. Integrating divides by 3, differentiating multiplies.

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2 Using identities

$\sin^2$, $\cos^2$, $\tan^2$

$$\cos^2x \equiv \tfrac{1}{2}(1 + \cos 2x)\ ;\ \sin^2x \equiv \tfrac{1}{2}(1 - \cos 2x)\ ;\ \tan^2x \equiv \sec^2x - 1$$ LEARN IT

from $\cos 2x$ and $1 + \tan^2x \equiv \sec^2x$

WORKED EXAMPLE

Find the exact value of $\int_0^{\pi/4} \cos^2x\, dx$.

1. $\tfrac{1}{2}\int_0^{\pi/4} (1 + \cos 2x)\, dx = \tfrac{1}{2}[x + \tfrac{1}{2}\sin 2x]_0^{\pi/4}$ M1 A1
2. $= \tfrac{1}{2}(\pi/4 + \tfrac{1}{2}) = \pi/8 + \tfrac{1}{4}$ A1

- **Products like sin x cos x:** use $\tfrac{1}{2}\sin 2x$, giving $-\tfrac{1}{4}\cos 2x$.

COMMON MISTAKE

Integrating $\sin^2x$ as $\sin^3x/3$. That is the chain rule backwards and only works if a cos x factor is present.

3 Partial fractions and f'/f

Splitting first

- **Factorised denominator?** Split into partial fractions (Algebra), then each piece gives a log.

WORKED EXAMPLE

Show that $\int_2^4 (5x + 1)/[(x - 1)(x + 2)]\, dx = \ln(243/8)$.

1. $(5x + 1)/[(x - 1)(x + 2)] = 2/(x - 1) + 3/(x + 2)$ B1
2. $[2\ln(x - 1) + 3\ln(x + 2)]_2^4 = (2\ln 3 + 3\ln 6) - (0 + 3\ln 4)$ M1 A1
3. $= \ln 9 + \ln(6/4)^3 = \ln(9 \times 27/8) = \ln(243/8)$ A1 AG

The f'(x)/f(x) pattern

$$\int f'(x)/f(x)\, dx = \ln|f(x)| + c$$

the top is (a multiple of) the derivative of the bottom

- **Examples:** $\int 2x/(x^2 + 1)\, dx = \ln(x^2 + 1)$; $\int \tan x\, dx = \int \sin x/\cos x\, dx = -\ln|\cos x| = \ln|\sec x|$.

WORKED EXAMPLE

Find the exact value of $\int_0^{\pi/3} \tan x\, dx$.

1. $[-\ln\cos x]_0^{\pi/3}$ M1
2. $= -\ln \tfrac{1}{2} + \ln 1 = \ln 2$ A1

COMMON MISTAKE

Using ln for $1/(x^2 + 1)$. The top must be the derivative of the bottom; $1/(x^2 + 1)$ gives $\tan^{-1}x$ instead.

4 Integration by parts

$\int u\,dv/dx\,dx = uv - \int v\,du/dx\,dx$

$$\int u\,(dv/dx)\,dx = uv - \int v\,(du/dx)\,dx$$ FORMULA LIST (MF19)

from the product rule, integrated

- **Choosing u:** pick the part that gets simpler when differentiated. Priority: ln x first, then powers of x, then e^x or trig.
- **ln x on its own:** write it as 1 × ln x with u = ln x, dv/dx = 1.
- **$x^2 e^x$** needs parts twice.

WORKED EXAMPLE

Find the exact value of $\int_0^{\pi/2} x \sin 2x\,dx$.

1. u = x, dv/dx = sin 2x ⇒ v = −½ cos 2x M1
2. = $[-\frac{1}{2}x\cos 2x]_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2}\cos 2x\,dx$ A1
3. = $-\frac{1}{2}(\pi/2)(-1) + \frac{1}{2}[\frac{1}{2}\sin 2x]_0^{\pi/2} = \pi/4 + 0$ M1
4. **= π/4** A1

WORKED EXAMPLE

Find $\int_1^e x^2 \ln x\,dx$.

1. u = ln x, dv/dx = x^2 ⇒ v = $x^3/3$ M1
2. = $[x^3 \ln x/3]_1^e - \int_1^e x^2/3\,dx$ A1
3. = $e^3/3 - [x^3/9]_1^e = e^3/3 - e^3/9 + 1/9$ M1
4. **= $(2e^3 + 1)/9$** A1

WORKED EXAMPLE

Find $\int_1^e \ln x\,dx$.

1. u = ln x, v = x: $[x \ln x]_1^e - \int_1^e 1\,dx$ M1
2. **= e − (e − 1) = 1** A1

COMMON MISTAKE

Sign errors with the minus in uv − ∫v du. Write the line out in full and bracket v du/dx.

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5 Integration by substitution

Change everything to u

- **Steps:** (1) write du/dx and make dx the subject; (2) replace x everywhere; (3) change the limits; (4) integrate in u. Never mix x and u in one integral.

WORKED EXAMPLE

Use the substitution $u = \sqrt{x}$ to find the exact value of $\int_1^4 1/(x + \sqrt{x})\,dx$.

1. $x = u^2$, $dx = 2u\,du$; limits $x = 1 \to u = 1$, $x = 4 \to u = 2$ M1
2. $\int_1^2 2u/(u^2 + u)\,du = \int_1^2 2/(u + 1)\,du$ A1
3. $= [2\ln(u + 1)]_1^2 = 2\ln 3 - 2\ln 2$ M1
4. **$= 2\ln(3/2)$** A1

WORKED EXAMPLE

Use $u = \sin x$ to find $\int_0^{\pi/2} \sin^3 x \cos x\,dx$.

1. $du = \cos x\,dx$; limits 0 and 1 M1
2. **$\int_0^1 u^3\,du = [u^4/4]_0^1 = \frac{1}{4}$** A1

COMMON MISTAKE

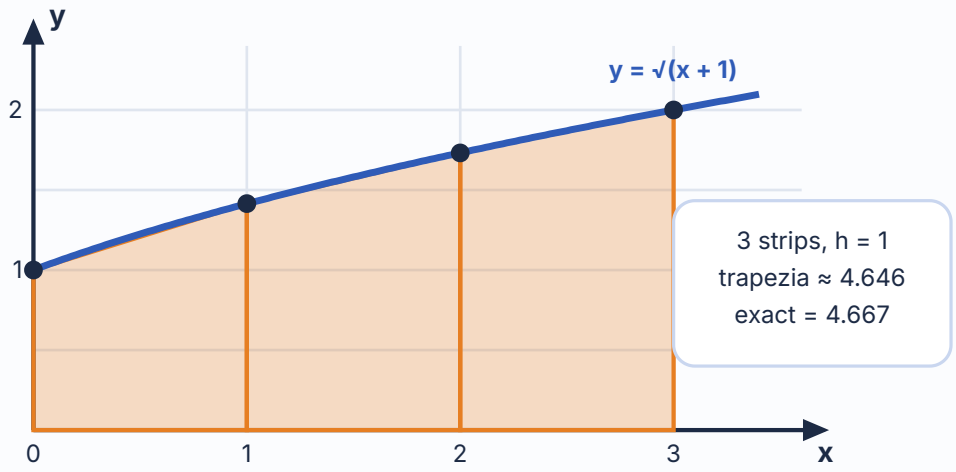
Not changing the limits, or leaving dx in place of 2u du. The 2024 Paper 3 report flagged both.

6 The trapezium rule

Estimating an area

$$\int_a^b y\,dx \approx (h/2)[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})],\ h = (b - a)/n$$ FORMULA LIST (MF19)

n strips means n + 1 ordinates



Straight tops on each strip: where the curve bends upwards, the trapezia sit above it.

- **Over or under?** Convex curve (bending upwards, like e^x) → **over-estimate**. Concave curve (bending downwards, like $\sqrt{x}$) → **under-estimate**. Justify with a sketch or the sign of d^2y/dx^2.

WORKED EXAMPLE

Use the trapezium rule with two intervals to estimate $\int_0^1 e^{x^2}\,dx$, and say whether it is an over- or under-estimate.

1. $h = 0.5$; $y = 1$, $e^{0.25} = 1.2840$, $e = 2.7183$ B1
2. $0.25 \times [1 + 2.7183 + 2(1.2840)] =$ **1.57** M1 A1
3. **the curve bends upwards on [0, 1], so it is an over-estimate** B1

COMMON MISTAKE

Counting strips as ordinates: 4 intervals need 5 y-values. And radians for any trig function.

LINKING QUESTION

The derivatives these reverse are in Further differentiation; Paper 1 integration and areas are in Integration; and these techniques solve Differential equations.

Test yourself — cover the answers first

QUICK CHECK

1. Integrate e^{-2x}.
2. Integrate $1/(9 + x^2)$.
3. Which method for $\int x e^x\,dx$?
4. Integrate $3x^2/(x^3 + 4)$.
5. Write $\sin^2 x$ in terms of $\cos 2x$.

EXAM-STYLE

6. Find the exact value of $\int_0^1 x e^{-x}\,dx$.
7. Find the exact value of $\int_0^{\pi/2} x \cos x\,dx$.
8. Show that $\int_0^{\ln 2} e^{3x}\,dx = 7/3$.

Show answers

1. $-\frac{1}{2}e^{-2x} + c$
2. $\frac{1}{3}\tan^{-1}(x/3) + c$
3. by parts, $u = x$
4. $\ln|x^3 + 4| + c$
5. $\frac{1}{2}(1 - \cos 2x)$
6. by parts: $[-x e^{-x}]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + 1 - e^{-1} = 1 - 2/e$
7. $[x \sin x]_0^{\pi/2} - \int \sin x\,dx = \pi/2 + [\cos x]_0^{\pi/2} = \pi/2 - 1$
8. $[\frac{1}{3}e^{3x}]_0^{\ln 2} = \frac{1}{3}(8 - 1) = 7/3$

Stuck on a question?

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