A LEVEL MATHS 9709 · PAPER 3 PURE MATHEMATICS 3 · 3.3 (ALSO PAPER 2) · ~8 MIN READ

Further trigonometry

sec, cosec and cot and their graphs, the identities $1 + \tan^2\theta \equiv \sec^2\theta$ and $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$, the addition and double-angle formulae, and expressing $a \sin\theta + b \cos\theta$ as $R \sin(\theta \pm \alpha)$ to solve equations and find maxima.

sec, cosec, cot · Pythagorean identities · Addition formulae · Double-angle formulae · Exact values · Proving identities · R sin(θ + α) · Solving equations

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Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **$\sec\theta = 1/\cos\theta$, $\operatorname{cosec}\theta = 1/\sin\theta$, $\cot\theta = 1/\tan\theta$.** Each has asymptotes where its parent function is zero.
- **Divide $\sin^2\theta + \cos^2\theta \equiv 1$ by $\cos^2\theta$** to get $1 + \tan^2\theta \equiv \sec^2\theta$; by $\sin^2\theta$ to get $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$.
- **Addition and double-angle formulae** are in the formula list; using them quickly is the skill. cos 2A has three forms: pick the one that leaves a single trig function.
- **$a \sin\theta + b \cos\theta = R \sin(\theta + \alpha)$** with $R = \sqrt{(a^2 + b^2)}$ and $\tan\alpha = b/a$. It turns a two-function equation into a one-function equation, and gives the maximum R and minimum −R.

BY THE END YOU CAN

- Use sec, cosec and cot and sketch their graphs.
- Use the Pythagorean, addition and double-angle identities to simplify, prove and evaluate exactly.
- Express $a \sin\theta + b \cos\theta$ in the forms $R \sin(\theta \pm \alpha)$ or $R \cos(\theta \pm \alpha)$.
- Solve trig equations in degrees or radians using these identities.
- Find maximum and minimum values of expressions using the R form.

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1 Reciprocal functions

sec, cosec and cot

Function	Definition	Undefined where	Range
sec θ	1/cos θ	cos θ = 0 (θ = 90°, 270°, ...)	sec θ ≤ −1 or sec θ ≥ 1
cosec θ	1/sin θ	sin θ = 0 (θ = 0°, 180°, ...)	cosec θ ≤ −1 or cosec θ ≥ 1
cot θ	1/tan θ = cos θ/sin θ	sin θ = 0	all real values

- **Graphs:** sec θ is ∪-shaped where cos θ > 0 and ∩-shaped where cos θ < 0, touching ±1 where cos θ = ±1. cosec θ is the same shape shifted by 90°.

$1 + \tan^2\theta \equiv \sec^2\theta$; $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$ **FORMULA LIST (MF19)**

divide $\sin^2\theta + \cos^2\theta \equiv 1$ by $\cos^2\theta$ or by $\sin^2\theta$

WORKED EXAMPLE

Solve $3\sec^2\theta - 5\tan\theta - 5 = 0$ for $0° \le \theta \le 360°$.

1. $3(1 + \tan^2\theta) - 5\tan\theta - 5 = 0 \Rightarrow 3\tan^2\theta - 5\tan\theta - 2 = 0$ M1
2. $(3\tan\theta + 1)(\tan\theta - 2) = 0$ M1
3. **$\tan\theta = 2 \Rightarrow \theta = 63.4°, 243.4°$; $\tan\theta = -\frac{1}{3} \Rightarrow \theta = 161.6°, 341.6°$** A1 A1

WORKED EXAMPLE

Solve $\tan\theta + \cot\theta = 4$ for $0° < \theta < 180°$.

1. $t + 1/t = 4 \Rightarrow t^2 - 4t + 1 = 0 \Rightarrow t = 2 \pm \sqrt{3}$ M1 A1
2. **$\theta = 75°$ or $\theta = 15°$** A1 **(tan 75° = 2 + √3 exactly)**

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2 Addition and double-angle formulae

The formulas

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$ FORMULA LIST (MF19)

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$ FORMULA LIST (MF19)

note the sign swaps

$$\tan(A \pm B) \equiv (\tan A \pm \tan B)/(1 \mp \tan A \tan B)$$ FORMULA LIST (MF19)

$$\sin 2A \equiv 2 \sin A \cos A \; ; \; \cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A \; ; \; \tan 2A \equiv 2 \tan A/(1 - \tan^2 A)$$

FORMULA LIST (MF19)

put B = A in the addition formulae

- **Choosing the cos 2A form:** if the equation has sin x, use $1 - 2\sin^2 x$; if it has cos x, use $2\cos^2 x - 1$.
- **For integration:** $\cos^2 x \equiv \frac{1}{2}(1 + \cos 2x)$ and $\sin^2 x \equiv \frac{1}{2}(1 - \cos 2x)$. See Further integration.

WORKED EXAMPLE

Find the exact value of cos 75°.

1. $\cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$ M1
2. $= (\sqrt{2}/2)(\sqrt{3}/2) - (\sqrt{2}/2)(\frac{1}{2})$ M1
3. **$= (\sqrt{6} - \sqrt{2})/4$** A1

WORKED EXAMPLE

Solve $\sin(x + 30°) = 2 \cos x$ for $0° \leq x \leq 360°$.

1. $\sin x \cos 30° + \cos x \sin 30° = 2 \cos x$ M1
2. $(\sqrt{3}/2) \sin x = (3/2) \cos x \Rightarrow \tan x = 3/\sqrt{3} = \sqrt{3}$ M1 A1
3. **$x = 60°, 240°$** A1

WORKED EXAMPLE

Solve $\cos 2x = 3 \sin x - 1$ for $0° \leq x \leq 360°$.

1. $1 - 2\sin^2 x = 3 \sin x - 1 \Rightarrow 2\sin^2 x + 3 \sin x - 2 = 0$ M1
2. $(2 \sin x - 1)(\sin x + 2) = 0$; $\sin x = -2$ is impossible M1
3. **$\sin x = \frac{1}{2} \Rightarrow x = 30°, 150°$** A1 A1

WORKED EXAMPLE

Prove that $\sin 2\theta/(1 + \cos 2\theta) \equiv \tan \theta$.

1. LHS $= 2 \sin \theta \cos \theta/(1 + 2\cos^2 \theta - 1)$ M1
2. **$= 2 \sin \theta \cos \theta/(2\cos^2 \theta) = \sin \theta/\cos \theta = \tan \theta =$ RHS** A1

COMMON MISTAKE

Writing cos 2θ as 2 cos θ, or sin 3α as 3 sin α. Only the formulae above are valid. The 2024 Paper 3 report also notes candidates who expanded sin(A + B) correctly but forgot to put tan 45° = 1 straight in.

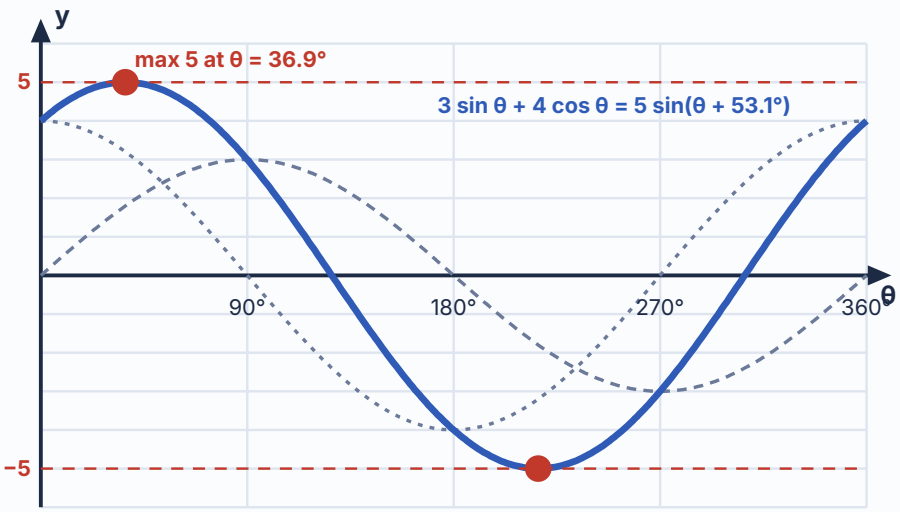
3 The R formula

a sin θ + b cos θ = R sin(θ + α)

- **Expand and compare:** R sin(θ + α) = R sin θ cos α + R cos θ sin α. Match coefficients: R cos α = a and R sin α = b.
- **So:** $R^2 = a^2 + b^2$ (square and add), and tan α = b/a (divide). Take α acute when a, b > 0.

$$a \sin\theta + b \cos\theta \equiv R \sin(\theta + \alpha),\ R = \sqrt{(a^2 + b^2)},\ \tan\alpha = b/a$$

similar forms R sin(θ − α), R cos(θ ± α): expand and compare the same way



Dashed: 3 sin θ and 4 cos θ. Their sum is a single sine wave of amplitude R = √(3² + 4²) = 5.

3 sin θ + 4 cos θ is a single sine wave with amplitude 5, shifted left by 53.1°.

WORKED EXAMPLE

(a) Express 3 sin θ + 4 cos θ in the form R sin(θ + α). (b) Solve 3 sin θ + 4 cos θ = 2 for 0° < θ < 360°. (c) Find the greatest value of 1/(7 − 3 sin θ − 4 cos θ).

1. (a) R = √(9 + 16) = 5; tan α = 4/3 ⇒ α = 53.13° M1 A1 A1
2. (b) sin(θ + 53.13°) = 0.4; θ + 53.13° lies between 53.13° and 413.13° M1
3. θ + 53.13° = 156.42° or 383.58° ⇒ **θ = 103.3°, 330.4°** A1 A1
4. **(c) 7 − 5 sin(θ + α) is least when the sine is 1: 7 − 5 = 2, so the greatest value is ½** B1

COMMON MISTAKE

Missing a solution because the interval for θ + α wasn't adjusted. Write the new interval first, as in (b).

LINKING QUESTION

Paper 1's basics are in Trigonometry; derivatives of trig functions are in Further differentiation; and trig substitutions and identities are used in Further integration.

Test yourself — cover the answers first

QUICK CHECK

1. Define cosec θ.
2. State the identity linking $\tan^2\theta$ and $\sec^2\theta$.
3. Write cos 2A in terms of sin A only.
4. Find R if $5\sin\theta - 12\cos\theta = R\sin(\theta - \alpha)$.
5. What is the minimum value of $3\sin\theta + 4\cos\theta$?

EXAM-STYLE

6. (a) Express $2\cos x - 5\sin x$ in the form $R\cos(x + \alpha)$, with $0° < \alpha < 90°$. (b) Hence solve $2\cos x - 5\sin x = 3$ for $0° < x < 360°$.
7. Prove that $\text{cosec}\,2\theta + \cot 2\theta \equiv \cot\theta$.
8. Solve $2\cos 2\theta + 5\cos\theta = 1$ for $0 \leq \theta \leq 2\pi$.

Show answers

1. $1/\sin\theta$
2. $1 + \tan^2\theta \equiv \sec^2\theta$
3. $1 - 2\sin^2 A$
4. 13
5. −5
6. (a) $\sqrt{29}\cos(x + 68.20°)$
   (b) $\cos(x + 68.20°) = 0.5571 \Rightarrow x + 68.20° = 303.85°$ or $416.15° \Rightarrow x = 235.7°, 347.9°$
7. $(1 + \cos 2\theta)/\sin 2\theta = 2\cos^2\theta/(2\sin\theta\cos\theta) = \cos\theta/\sin\theta = \cot\theta$
8. $4\cos^2\theta + 5\cos\theta - 3 = 0 \Rightarrow \cos\theta = (-5 + \sqrt{73})/8 = 0.443 \Rightarrow \theta = 1.11, 5.17$

Stuck on a question?

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