A LEVEL MATHS 9709 · PAPER 1 PURE MATHEMATICS 1 · 1.8 · ~8 MIN READ

Integration

Integration as reverse differentiation, integrating $(ax + b)^n$, finding the constant from a point, definite and improper integrals, areas between curves and lines, and volumes of revolution about either axis.

Reverse differentiation | $\int(ax + b)^n\,dx$ | Constant of integration | Definite integrals | Improper integrals | Area under a curve | Area between curves | Volumes of revolution

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Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **Integration reverses differentiation.** $\int x^n\,dx = x^{n+1}/(n + 1) + c$ for $n \neq -1$: add one to the power, divide by the new power, add c.
- **For $(ax + b)^n$**, also divide by a: $\int(ax + b)^n\,dx = (ax + b)^{n+1}/[a(n + 1)] + c$.
- **A definite integral** gives the signed area between a curve and the x-axis. Area below the axis comes out negative, so split the integral where the curve crosses the axis.
- **Area between two curves** = ∫(top − bottom) dx between their intersections.
- **Volume of revolution** about the x-axis: $V = \pi\int y^2\,dx$. About the y-axis: $V = \pi\int x^2\,dy$.

BY THE END YOU CAN

- Integrate sums of powers of x and expressions of the form $(ax + b)^n$.
- Find the equation of a curve from its gradient function and a point.
- Evaluate definite integrals, including simple improper integrals.
- Find areas under curves, between a curve and a line, and between two curves.
- Find volumes of revolution about the x- and y-axes, including regions not bounded by the axis.

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1 Indefinite integrals

Reverse the power rule

$$\int x^n \, dx = x^{n+1}/(n+1) + c, \; n \neq -1$$ FORMULA LIST (MF19)

c is the constant of integration

$$\int (ax+b)^n \, dx = (ax+b)^{n+1} / [a(n+1)] + c$$

$n \neq -1$; check by differentiating with the chain rule

- **Why the "+ c":** many functions have the same derivative ($x^2 + 1$ and $x^2 - 7$ both differentiate to $2x$). The constant is fixed only when you know a point on the curve.
- **Rewrite as powers first,** exactly as when differentiating.

WORKED EXAMPLE

Find $\int (6x^2 - 4/x^3 + \sqrt{x})\, dx$.

1. $= \int (6x^2 - 4x^{-3} + x^{1/2})\, dx$ M1
2. **$= 2x^3 + 2x^{-2} + (2/3)x^{3/2} + c$** A1 A1

WORKED EXAMPLE

Find $\int (2x+1)^4 \, dx$ and $\int 3/\sqrt{(4x-3)}\, dx$.

1. $(2x+1)^5/(2 \times 5) + c =$ **$(2x+1)^5/10 + c$** B1
2. $3\int (4x-3)^{-1/2} \, dx = 3 \times (4x-3)^{1/2}/(4 \times \tfrac{1}{2})$ M1
3. **$= (3/2)\sqrt{(4x-3)} + c$** A1

COMMON MISTAKE

Multiplying by a instead of dividing. The derivative of $(ax+b)^{n+1}$ brings out a factor a, so the integral must divide by a. Check by differentiating your answer.

Finding the constant

WORKED EXAMPLE

A curve has gradient $dy/dx = 3x^2 - 4x + 1$ and passes through (2, 5). Find its equation.

1. $y = x^3 - 2x^2 + x + c$ M1 A1
2. $5 = 8 - 8 + 2 + c \Rightarrow c = 3$ M1
3. **$y = x^3 - 2x^2 + x + 3$** A1

COMMON MISTAKE

Leaving out c, or setting y = 0. The 2024 report noted candidates who omitted c or ignored the given condition $f(1) = -1$. Substitute the given point.

2 Definite integrals

Evaluating

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

F is any integral of f; c cancels

- **Show the substitution:** write F(b) − F(a) with numbers in before simplifying. Examiners want to see the limits substituted.
- **Improper integrals:** one limit is infinite, or the function is undefined at a limit. Evaluate with a general limit and see what happens as it tends to ∞ or 0.

WORKED EXAMPLE

Evaluate $\int_1^{\infty} 6/x^3\,dx$ and $\int_0^9 1/\sqrt{x}\,dx$.

1. $[-3x^{-2}]_1^{\infty} = 0 - (-3) = \mathbf{3}$ (since $-3/x^2 \to 0$ as $x \to \infty$) M1 A1
2. **$[2x^{1/2}]_0^9 = 2 \times 3 - 0 = 6$** M1 A1

3 Areas

Under a curve, and below the axis

- **Area between y = f(x), the x-axis and x = a to x = b** is $\int_a^b y\,dx$ when the curve is above the axis.
- **Below the axis** the integral is negative. Find where the curve crosses the axis and integrate each part separately, then add the sizes.

WORKED EXAMPLE

Find the total area between $y = x^2 - 2x$, the x-axis, x = 0 and x = 3.

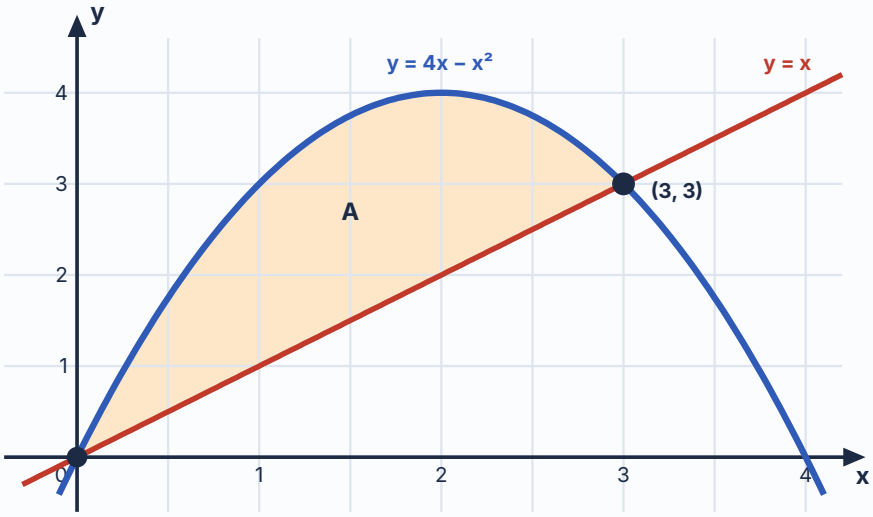
1. the curve crosses the axis at x = 0 and x = 2 B1
2. $\int_0^2 (x^2 - 2x)\,dx = [x^3/3 - x^2]_0^2 = 8/3 - 4 = -4/3 \Rightarrow$ area 4/3 M1
3. $\int_2^3 (x^2 - 2x)\,dx = 0 - (-4/3) = 4/3$ M1
4. **total area = 8/3** A1 **(but $\int_0^3 = 0$, because the two parts cancel)**

Between two curves

$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx$$

f is the upper curve, a and b are the x-coordinates of the intersections

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$A = \int_0^3 [(4x - x^2) - x]\,dx = \int_0^3 (3x - x^2)\,dx = 9/2$. Top curve minus bottom curve.

Subtract the lower curve from the upper one, then integrate between the intersection points.

WORKED EXAMPLE

Find the area enclosed by $y = 4x - x^2$ and $y = x$.

1. intersections: $4x - x^2 = x \Rightarrow x(3 - x) = 0 \Rightarrow x = 0, 3$ M1 A1
2. area $= \int_0^3 [(4x - x^2) - x]\,dx = \int_0^3 (3x - x^2)\,dx$ M1
3. $= [3x^2/2 - x^3/3]_0^3 = 27/2 - 9 = 9/2$ A1

COMMON MISTAKE

Combining several pieces into one integral with the wrong sign. In 2024 this caused sign errors. Write each area as its own integral with clear limits, then add or subtract.

- **Area between a curve and the y-axis:** rearrange to $x = g(y)$ and use $\int x\,dy$ between y-limits.

4 Volumes of revolution

Discs and washers

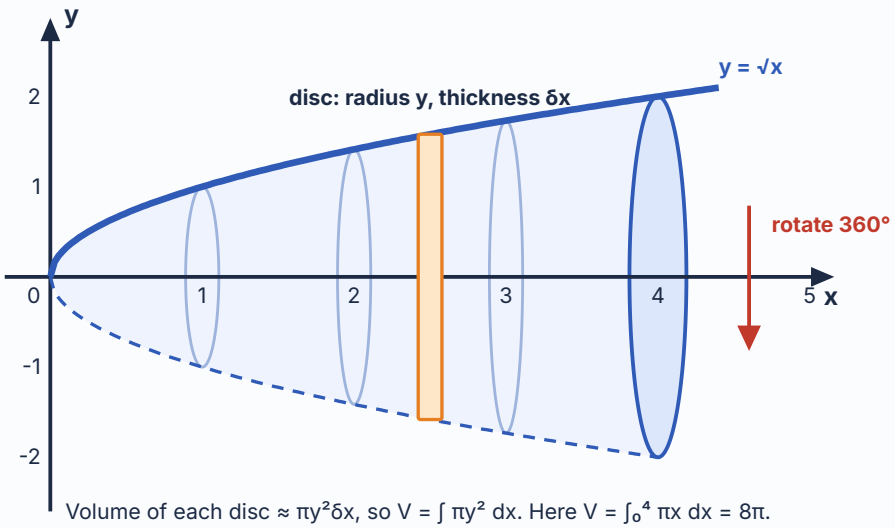
$$V = \pi\int_a^b y^2\,dx$$

rotating 360° about the x-axis

$$V = \pi\int_p^q x^2\,dy$$

rotating 360° about the y-axis

- **Why:** a thin slice of thickness δx is a disc of radius y, so its volume is about $\pi y^2 \delta x$. Adding the slices and letting $\delta x \to 0$ gives the integral.
- **Region not touching the axis (a washer):** $V = \pi\int(y_{outer}^2 - y_{inner}^2)\,dx$. Square each radius separately; don't square the difference.



Volume of each disc ≈ $\pi y^2 \delta x$, so $V = \int \pi y^2 \, dx$. Here $V = \int_0^4 \pi x \, dx = 8\pi$.

Rotating $y = \sqrt{x}$ about the x-axis: each slice is a disc of radius $\sqrt{x}$.

WORKED EXAMPLE

The region bounded by $y = \sqrt{x}$, the x-axis and $x = 4$ is rotated through 360° about the x-axis. Find the volume.

1. $V = \pi\int_0^4 (\sqrt{x})^2 \, dx = \pi\int_0^4 x \, dx$ M1
2. $= \pi[x^2/2]_0^4 = 8\pi$ A1

WORKED EXAMPLE

The region between $y = 2x$ and $y = x^2$ is rotated through 360° about the x-axis. Find the exact volume.

1. intersections at $x = 0$ and $x = 2$; for $0 < x < 2$, $2x$ is the outer curve B1
2. $V = \pi\int_0^2 [(2x)^2 - (x^2)^2] \, dx = \pi\int_0^2 (4x^2 - x^4) \, dx$ M1
3. $= \pi[4x^3/3 - x^5/5]_0^2 = \pi(32/3 - 32/5) = 64\pi/15$ A1

WORKED EXAMPLE

The part of $y = x^2$ between $y = 0$ and $y = 4$ is rotated about the y-axis. Find the volume.

1. $x^2 = y$, so $V = \pi\int_0^4 y \, dy$ M1
2. $= \pi[y^2/2]_0^4 = 8\pi$ A1

COMMON MISTAKE

Using $\pi\int(y_1 - y_2)^2 \, dx$ for a washer. That is the volume of a different solid. It must be $\pi\int(y_1^2 - y_2^2) \, dx$.

LINKING QUESTION

Integration reverses Differentiation. Paper 3 adds exponentials, logs, trig, partial fractions, parts and substitution in Further integration, and kinematics uses integration to get displacement from velocity in Kinematics in a straight line.

Test yourself — cover the answers first

QUICK CHECK

1. Find $\int(x^3 + 2) \, dx$.
2. Find $\int(5 - 2x)^3 \, dx$.
3. Evaluate $\int_1^4 \sqrt{x} \, dx$.
4. Write the formula for the volume when $y = f(x)$ is rotated about the x-axis.

EXAM-STYLE

5. The curve $y = 9 - x^2$ meets the x-axis at A and B. (a) Find the area between the curve and the x-axis. (b) Find the volume when this region is rotated about the x-axis.

6. A curve has $dy/dx = 6/(2x - 1)^2$ and passes through (1, 1). Find its equation.

7. Show that $\int_1^{\infty} x^{-3/2}\,dx$ is finite and find its value.

Show answers

1. $x^4/4 + 2x + c$
2. $-(5 - 2x)^4/8 + c$
3. 14/3
4. $V = \pi\int y^2\,dx$
5. (a) $\int_{-3}^{3} (9 - x^2)\,dx = 36$
   (b) $\pi\int_{-3}^{3} (9 - x^2)^2\,dx = 1296\pi/5$
6. $y = -3/(2x - 1) + c$; $1 = -3 + c \Rightarrow c = 4$; $y = 4 - 3/(2x - 1)$
7. $[-2x^{-1/2}]_1^{\infty} = 0 + 2 = 2$

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