A LEVEL MATHS 9709 · PAPER 3 PURE MATHEMATICS 3 · 3.2 (ALSO PAPER 2) · ~8 MIN READ

Logarithms and exponential functions

Logarithms as inverses of indices, the laws of logarithms, e^x and ln x and their graphs, solving equations and inequalities with the unknown in an index, and turning $y = kx^n$ and $y = ka^x$ into straight lines to find constants.

Logs & indices | Laws of logarithms | e^x and ln x | Graphs of e^{kx} | Index equations | Inequalities | Hidden quadratics | Linearising with logs

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Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **A logarithm is an index:** $\log_a b = c$ means $a^c = b$. ln x is $\log_e x$, so ln and e are inverse functions.
- **Three laws:** $\log(xy) = \log x + \log y$; $\log(x/y) = \log x - \log y$; $\log(x^n) = n \log x$. There is no law for $\log(x + y)$.
- **Unknown in an index?** Take ln of both sides and bring the power down.
- **Straight-line graphs from curves:** $y = kx^n$ gives $\ln y = \ln k + n \ln x$ (plot ln y against ln x); $y = ka^x$ gives $\ln y = \ln k + x \ln a$ (plot ln y against x).

BY THE END YOU CAN

- Relate logarithms to indices and use the laws of logarithms.
- Use the properties and graphs of e^x and ln x, including $y = e^{kx}$ for positive and negative k.
- Solve equations and inequalities with the unknown in an index.
- Reduce $y = kx^n$ and $y = ka^x$ to linear form and find the constants from a graph.

1 Logarithms

Definition and laws

Logarithm — $\log_a b$ is the power to which a must be raised to give b: $\log_a b = c \Leftrightarrow a^c = b$ ($a > 0$, $a \neq 1$, $b > 0$).

$$\log(xy) = \log x + \log y \ ; \ \log(x/y) = \log x - \log y \ ; \ \log(x^n) = n \log x$$ LEARN IT

any base, x, y > 0

- **Why the laws hold:** they are the index laws written backwards. If $x = a^p$ and $y = a^q$, then $xy = a^{p+q}$, so $\log xy = p + q$.
- **Special values:** $\log_a 1 = 0$, $\log_a a = 1$, $\ln 1 = 0$, $\ln e = 1$, $e^{\ln x} = x$, $\ln(e^x) = x$.

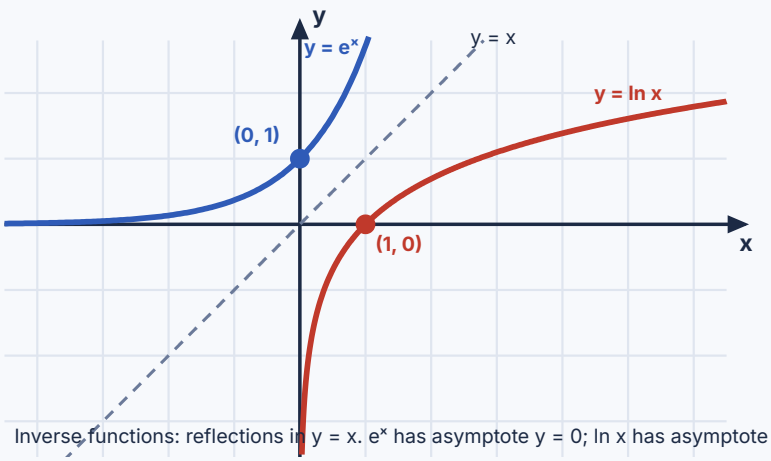
COMMON MISTAKE

log(x + y) = log x + log y is false. So is (log x)/(log y) = log(x − y). Combine logs only with the three laws.

COMMON MISTAKE

Losing brackets: $3 + 2 \ln 5x$ is not $\ln(3 \times (5x)^2)$. The June 2024 Paper 2 report found many errors like this. Write each step separately.

e^x and ln x



Inverse functions: reflections in y = x. e^x has asymptote y = 0; ln x has asymptote x = 0.

$y = e^x$ and $y = \ln x$ are reflections of each other in $y = x$.

	$y = e^x$	$y = \ln x$
Domain	all real x	$x > 0$
Range	$y > 0$	all real y
Key point	(0, 1)	(1, 0)
Asymptote	$y = 0$ (as $x \to -\infty$)	$x = 0$ (as $x \to 0^+$)

- **$y = e^{kx}$:** $k > 0$ gives growth; $k < 0$ gives decay towards $y = 0$ (e.g. e^{-2x}). Both pass through (0, 1).
- **Transformations** apply as usual: $y = 3 - e^{-x}$ has asymptote $y = 3$ and y-intercept 2.

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2 Solving equations and inequalities

Unknown in an index

WORKED EXAMPLE

Solve $5^{2x-1} = 8$, giving x to 3 s.f.

1. $(2x - 1) \ln 5 = \ln 8$ M1
2. $2x - 1 = \ln 8/\ln 5 = 1.292$ A1
3. **$x = 1.15$** A1

WORKED EXAMPLE

Solve $3^{x+1} = 4^x$.

1. $(x + 1) \ln 3 = x \ln 4$ M1
2. $x(\ln 4 - \ln 3) = \ln 3 \Rightarrow x = \ln 3/(\ln 4 - \ln 3)$ M1
3. **$x = 3.82$** A1

WORKED EXAMPLE

Solve $2^{2x} - 5(2^x) + 6 = 0$.

1. let $u = 2^x$: $u^2 - 5u + 6 = 0 \Rightarrow u = 2$ or $u = 3$ M1 A1
2. $2^x = 2 \Rightarrow x = 1$; $2^x = 3 \Rightarrow x = \ln 3/\ln 2 = 1.585$ A1
3. **$x = 1$ or $x = 1.58$**

WORKED EXAMPLE

Solve $e^{2x} - 4e^x - 5 = 0$.

1. $(e^x - 5)(e^x + 1) = 0$ M1
2. $e^x = -1$ has no solution ($e^x > 0$) B1
3. **$x = \ln 5$** A1

Log equations

WORKED EXAMPLE

Solve $\ln(x + 1) + \ln(x - 1) = \ln 3$.

1. $\ln[(x + 1)(x - 1)] = \ln 3 \Rightarrow x^2 - 1 = 3$ M1
2. $x = \pm 2$ A1
3. **$x - 1$ must be positive, so $x = 2$ (reject -2)** A1

COMMON MISTAKE

Not rejecting a root that makes a logarithm's argument zero or negative. The 2024 Paper 3 reports flagged this.

Inequalities

- **Dividing by a negative log reverses the inequality.** ln a < 0 when 0 < a < 1.

WORKED EXAMPLE

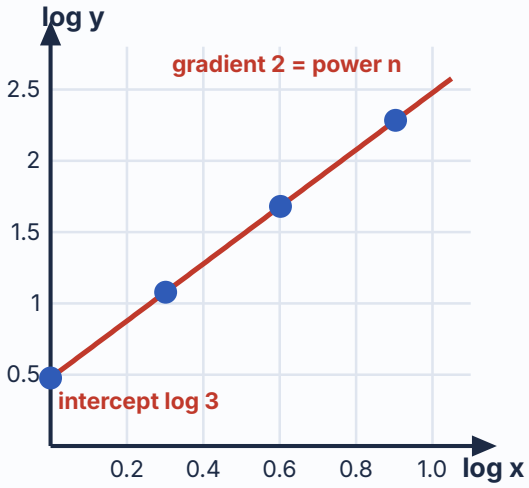
Find the smallest integer n for which $0.8^n < 0.1$.

1. n ln 0.8 < ln 0.1 M1
2. ln 0.8 < 0, so dividing reverses the sign: n > ln 0.1/ln 0.8 = 10.3 M1
3. **n = 11** A1

3 Linearising with logarithms

Two models

Model	Take ln	Plot	Gradient	Intercept
$y = kx^n$	ln y = ln k + n ln x	ln y against ln x	n	ln k
$y = ka^x$	ln y = ln k + x ln a	ln y against x	ln a	ln k



straight on (log x, log y)
⇒ y = ax^n
here $y = 3x^2$

A power law becomes a straight line on a graph of log y against log x.

- **Finding constants:** read two points on the line (or use the gradient and intercept), then undo the logs: $k = e^{\text{intercept}}$, $a = e^{\text{gradient}}$.
- **Name the variables in the straight-line form:** "ln y = 0.8 + 2.6 ln x", then convert.

WORKED EXAMPLE

The graph of ln y against ln x is a straight line through (0.5, 2.1) and (2.5, 7.3). Find y in terms of x.

1. gradient $n = (7.3 - 2.1)/(2.5 - 0.5) = 2.6$ M1
2. intercept: $2.1 = \ln k + 2.6 \times 0.5 \Rightarrow \ln k = 0.8$ M1
3. **$k = e^{0.8} = 2.23$: $y = 2.23x^{2.6}$** A1

WORKED EXAMPLE

The graph of ln y against x is a straight line through (1, 1.9) and (4, 3.4). Find $y = A \times b^x$.

1. gradient $\ln b = 1.5/3 = 0.5 \Rightarrow b = e^{0.5} = 1.65$ M1 A1
2. $\ln A = 1.9 - 0.5 = 1.4 \Rightarrow A = e^{1.4} = 4.06$ A1
3. **$y = 4.06 \times 1.65^x$**

COMMON MISTAKE

Substituting ln x values as x values when reading a log graph. In 2024 this stopped many candidates. If the axis is ln x, the point (0.5, 2.1) means ln x = 0.5, not x = 0.5.

LINKING QUESTION

e^x and ln x are differentiated and integrated in Further differentiation and Further integration; exponential growth and decay come from Differential equations; and inverse functions are explained in Functions and transformations.

Test yourself — cover the answers first

QUICK CHECK

1. Write $\log_2 32$ as an integer.
2. Simplify $\ln(e^3)$.
3. Solve $e^x = 7$.
4. Which graph is straight for $y = 3 \times 2^x$?
5. Express $2 \ln 3 - \ln 6$ as a single logarithm.

EXAM-STYLE

6. Solve $2 \ln x - \ln(x + 2) = \ln 3$.
7. The variables x and y satisfy $y = Ax^n$. When ln y is plotted against ln x the line has gradient −1.5 and passes through (2, 0.4). Find A and n.
8. Solve the inequality $3^x > 2^{x+2}$.

Show answers

1. 5
2. 3
3. $x = \ln 7$
4. $\ln y$ against x
5. $\ln(9/6) = \ln 1.5$
6. $\ln(x^2/(x + 2)) = \ln 3 \Rightarrow x^2 - 3x - 6 = 0 \Rightarrow x = (3 + \sqrt{33})/2 = 4.37$ (reject the negative root)
7. $n = -1.5$; $\ln A = 0.4 + 3 = 3.4 \Rightarrow A = 30.0$
8. $x \ln 3 > (x + 2) \ln 2 \Rightarrow x(\ln 3 - \ln 2) > 2 \ln 2 \Rightarrow x > 3.42$

Stuck on a question?

Send it to me on WhatsApp. I read every message.

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