A LEVEL MATHS 9709 · PAPER 3 PURE MATHEMATICS 3 · 3.6 (ALSO PAPER 2) · ~6 MIN READ

Numerical solution of equations

Locating a root by a change of sign or by sketching two graphs, rearranging an equation into x = F(x), running the iteration $x_{n+1} = F(x_n)$ to a stated accuracy, and understanding why some rearrangements converge and others do not.

Change of sign | Sketching to count roots | Rearranging to x = F(x) | Iteration | Staircase & cobweb diagrams | Convergence | Stating accuracy

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Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **Change of sign:** if f(x) is continuous and f(a) and f(b) have opposite signs, there is a root between a and b.
- **Iteration:** rearrange f(x) = 0 into x = F(x), pick x_1, and keep calculating $x_{n+1} = F(x_n)$. If the values settle, the limit is a root.
- **Convergence needs $|F'(x)| < 1$ near the root.** A different rearrangement of the same equation may run away instead.
- **Accuracy:** to give a root to 2 d.p., iterate until two successive values agree to 2 d.p. (show 4 d.p. in working), or check the sign change across the interval ±0.005.

BY THE END YOU CAN

- Locate roots graphically or by a sign change, and know when a sign change can mislead.
- Show that a root satisfies a given rearranged equation.
- Use an iterative formula to find a root to a given accuracy, showing all iterates.
- Understand convergence and divergence using staircase and cobweb diagrams.

1 Locating roots

Sign change and sketches

- **Sketch two graphs:** to count the roots of $e^x = 3 - x$, sketch $y = e^x$ and $y = 3 - x$. Each crossing is a root. Questions often say "by sketching a suitable pair of graphs".
- **Sign change:** evaluate f at both ends and state the signs clearly, with a conclusion sentence: "the sign changes and f is continuous, so there is a root in (a, b)".
- **When it fails:** a vertical asymptote (e.g. tan x or 1/(x − 2)) gives a sign change with no root; two roots close together give no sign change at all.

WORKED EXAMPLE

Show that $x^3 + 2x - 5 = 0$ has a root between 1 and 2.

1. f(x) = $x^3 + 2x - 5$; f(1) = −2 < 0, f(2) = 7 > 0 M1
2. **sign change and f is continuous, so a root lies between 1 and 2** A1

COMMON MISTAKE

Missing the conclusion, or writing "f(1) = −2, f(2) = 7" with no comment. The mark needs the sign change and a statement.

2 Iteration

$x_{n+1} = F(x_n)$

- **Show that a root satisfies x = F(x):** start from the original equation and rearrange step by step to the given form. For example $x^3 + 2x - 5 = 0 \Rightarrow x^3 = 5 - 2x \Rightarrow x = \sqrt[3]{(5 - 2x)}$.
- **Calculator routine:** type x_1, press =, then type F(Ans) and press = repeatedly. Write down every iterate to 4 or more d.p.

WORKED EXAMPLE

Use $x_{n+1} = \sqrt[3]{(5 - 2x_n)}$ with $x_1 = 1.3$ to find the root to 3 d.p.

1. $x_2 = 1.33887$, $x_3 = 1.32425$, $x_4 = 1.32978$, $x_5 = 1.32770$, $x_6 = 1.32849$, $x_7 = 1.32819$, $x_8 = 1.32830$, $x_9 = 1.32826$ M1 A1
2. **values settle at 1.328 (3 d.p.)** A1

WORKED EXAMPLE

The stationary point of $y = x \cos x$ for $0 < x < \pi/2$ has x-coordinate p. Show that $p = \tan^{-1}(1/p)$, and use an iteration to find p to 2 d.p.

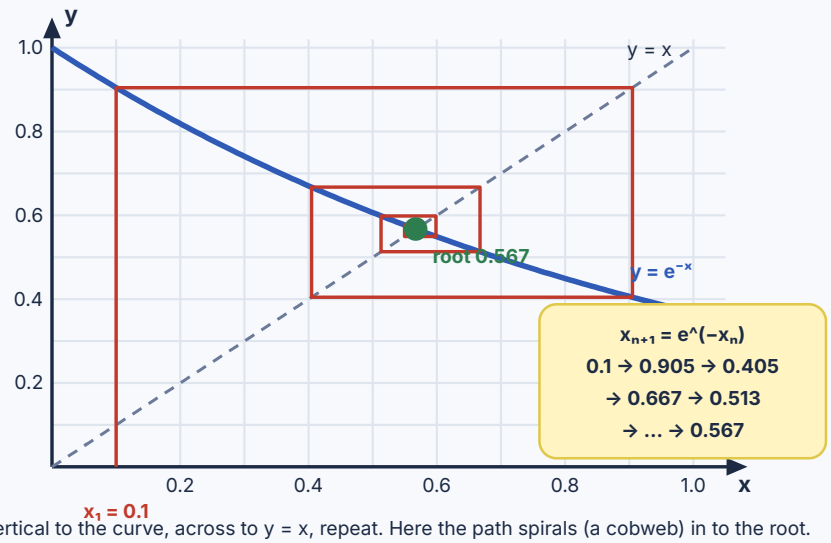
1. $dy/dx = \cos x - x \sin x = 0 \Rightarrow \tan x = 1/x \Rightarrow x = \tan^{-1}(1/x)$ M1 A1
2. $x_1 = 1$: 0.7854, 0.9050, 0.8352, 0.8749, 0.8520, 0.8651, 0.8576, 0.8619, 0.8594, 0.8609, ... M1
3. **p = 0.86** A1 **(check: f(x) = tan x – 1/x changes sign between 0.855 and 0.865)**

COMMON MISTAKE

Stopping too early. Two iterates that round to the same value to the required accuracy are the minimum evidence. The 2024 Paper 3 report noted iterates given to only 2 d.p.

3 Why iterations converge

Staircases and cobwebs



Vertical to the curve, across to y = x, repeat. Here the path spirals (a cobweb) in to the root.

For $x_{n+1} = e^{-x_n}$ the gradient of F is negative, so the iterates spiral in (a cobweb) towards 0.567.

- **Picture:** draw y = x and y = F(x). Go up to the curve, across to y = x, up to the curve again. The root is where they cross.
- **Staircase:** when $0 < F'(x) < 1$, the values approach the root from one side. **Cobweb:** when $-1 < F'(x) < 0$, they alternate either side.
- **Divergence:** when $|F'(x)| > 1$ near the root, the iterates move away. Try a different rearrangement.

WORKED EXAMPLE

The equation $x = 2 + \ln x$ has a root near 3. Compare the iterations $x_{n+1} = 2 + \ln x_n$ and $x_{n+1} = e^{x_n - 2}$.

1. first: $F'(x) = 1/x \approx 0.32$ near 3.15, so it converges: 3, 3.0986, 3.1310, 3.1413, 3.1446, 3.1457, 3.1460, ... → **3.146** M1 A1
2. **second: $F'(x) = e^{x-2} \approx 3.15 > 1$, so it diverges: 3.1, 3.0042, 2.7296, 2.0743, 1.0771, ... moving away** A1

COMMON MISTAKE

Claiming an iteration "doesn't work" because the root is wrong. Both forms have the same root; only the convergence differs.

LINKING QUESTION

The functions you iterate come from Logarithms and exponentials and Further trigonometry; stationary-point equations come from Further differentiation; and the trapezium rule is in Further integration.

Test yourself — cover the answers first

QUICK CHECK

1. $f(2) = -0.3$ and $f(3) = 1.4$. What can you conclude if f is continuous?
2. Why might $f(x) = 1/(x - 2)$ give a sign change on (1, 3)?
3. What condition makes $x = F(x)$ converge?
4. Iterates alternate either side of the root. What diagram is this?
5. Rearrange $x^2 - 3x - 1 = 0$ into $x = F(x)$ with a square root.

EXAM-STYLE

6. Use $x_{n+1} = e^{-x_n}$ with $x_1 = 0.5$ to find the root of $x = e^{-x}$ to 2 d.p.
7. Show that $x = 2 + \ln x$ has a root between 3 and 4.
8. Explain why $x_{n+1} = e^{x_n - 2}$ cannot find the root near 3.15 of $x = 2 + \ln x$.

Show answers

1. there is a root between 2 and 3
2. asymptote at x = 2, not a root
3. $|F'(x)| < 1$ near the root
4. a cobweb
5. $x = \sqrt{(3x + 1)}$
6. 0.6065, 0.5452, 0.5797, 0.5601, 0.5712, 0.5649, ... → 0.567, so 0.57
7. $f(x) = 2 + \ln x - x$: $f(3) = 0.099 > 0$, $f(4) = -0.614 < 0$; sign change, continuous, so a root
8. $F'(x) = e^{x-2} \approx 3.1 > 1$ there, so the iterates diverge

Stuck on a question?

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