A LEVEL MATHS 9709 · PAPER 1 PURE MATHEMATICS 1 · 1.1 · ~13 MIN READ

Quadratics

Completing the square, the discriminant, quadratic inequalities, line–curve intersections and equations that are quadratics in disguise, with the working Cambridge examiners expect to see.

Completing the square | Vertex & range | Discriminant | Solving quadratics | Quadratic inequalities | Line meets curve | Quadratics in disguise

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Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **Completing the square** turns $ax^2 + bx + c$ into $a(x - p)^2 + q$. It gives the vertex (p, q), the axis of symmetry x = p, and the greatest or least value of the expression.
- **The discriminant $b^2 - 4ac$** tells you how many real roots there are: two if it is positive, one repeated root if zero, none if negative.
- **Inequalities:** find the critical values, sketch the parabola, then read off the region. Never divide an inequality by something whose sign you don't know.
- **Line meets curve:** substitute to get one quadratic. Its discriminant decides whether the line cuts, touches (tangent) or misses the curve.
- **Show your working.** Cambridge gives no marks for unsupported calculator answers, so factorise or substitute into the formula on paper.

BY THE END YOU CAN

- Complete the square for any quadratic, including $a \neq 1$, and use it to find a vertex or a range.
- Use the discriminant to find conditions on a constant k.
- Solve quadratic inequalities and write the answer correctly.
- Solve a linear and a quadratic equation simultaneously and decide whether a line is a tangent.
- Recognise and solve equations that are quadratic in x^2, $\sqrt{x}$, tan x and similar.

1 Completing the square

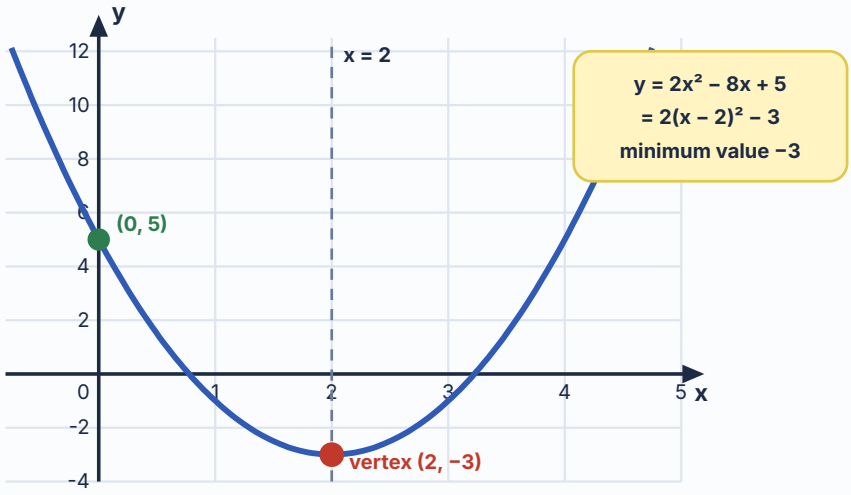
Why it works

- $x^2 + bx$ is the start of a perfect square: $(x + b/2)^2 = x^2 + bx + (b/2)^2$. So $x^2 + bx = (x + b/2)^2 - (b/2)^2$.
- **Halve the x-coefficient, square it, subtract it.** Everything else is bookkeeping.
- With $a \neq 1$, take a out of the x^2 and x terms first: $ax^2 + bx + c = a[x^2 + (b/a)x] + c$.

$$ax^2 + bx + c = a(x - p)^2 + q$$

vertex (p, q); axis of symmetry x = p

- **a > 0:** the parabola is ∪-shaped, so q is the **minimum** value. **a < 0:** ∩-shaped, so q is the **maximum**.



Completed square $a(x - p)^2 + q$: vertex (p, q), axis of symmetry $x = p$.

The completed square shows the vertex directly. Here the minimum value of the expression is −3, when $x = 2$.

Worked examples

WORKED EXAMPLE

Express $2x^2 - 12x + 13$ in the form $a(x + b)^2 + c$, and state the range of $f(x) = 2x^2 - 12x + 13$ for $x \in \mathbb{R}$.

1. $2x^2 - 12x + 13 = 2[x^2 - 6x] + 13$ M1
2. $= 2[(x - 3)^2 - 9] + 13 = 2(x - 3)^2 - 18 + 13$
3. $= \mathbf{2(x - 3)^2 - 5}$ A1 A1 ($a = 2$, $b = -3$, $c = -5$)
4. **$(x - 3)^2 \geq 0$, so $f(x) \geq -5$. Range: $f(x) \geq -5$** B1

WORKED EXAMPLE

Express $5 + 8x - 2x^2$ in the form $p - q(x - r)^2$ and hence state the maximum value.

1. $5 + 8x - 2x^2 = 5 - 2[x^2 - 4x]$ M1
2. $= 5 - 2[(x - 2)^2 - 4] = \mathbf{13 - 2(x - 2)^2}$ A1
3. **the squared term is never negative, so the maximum is 13, when $x = 2$** B1

COMMON MISTAKE

Not checking your answer. The June 2024 Paper 1 report says many candidates did not check that their completed square expanded back to the original. Expand it in your head: $2(x - 3)^2 - 5 = 2x^2 - 12x + 18 - 5$ ✓.

COMMON MISTAKE

Forgetting to multiply the $-(b/2)^2$ by a. In $2[(x - 3)^2 - 9] + 13$, the -9 becomes -18, not -9.

2 The discriminant

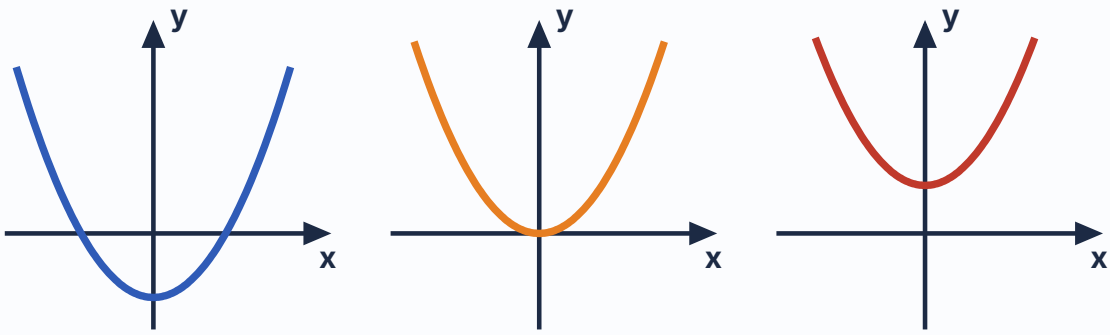
Three cases

$b^2 - 4ac$ **LEARN IT**

the expression under the square root in the quadratic formula

Discriminant	Roots of $ax^2 + bx + c = 0$	Graph of $y = ax^2 + bx + c$
$b^2 - 4ac > 0$	two distinct real roots	crosses the x-axis twice
$b^2 - 4ac = 0$	one repeated (equal) root	touches the x-axis (vertex on the axis)
$b^2 - 4ac < 0$	no real roots	never meets the x-axis
$b^2 - 4ac \geq 0$	real roots (equal or distinct)	meets the x-axis

The discriminant counts where $y = ax^2 + bx + c$ meets the x-axis



$b^2 - 4ac > 0$
two real roots

$b^2 - 4ac = 0$
one repeated root

$b^2 - 4ac < 0$
no real roots

Shifting the same parabola up changes the discriminant from positive to zero to negative.

- **Why:** the formula $x = [-b \pm \sqrt{(b^2 - 4ac)}]/2a$ has two different values if the square root is of a positive number, one value if it is of zero, and no real value if it is of a negative number.
- **"Positive for all x"** (always above the axis) means **$a > 0$ and $b^2 - 4ac < 0$**. "Negative for all x" means $a < 0$ and $b^2 - 4ac < 0$.

Conditions on a constant k

WORKED EXAMPLE

Find the set of values of k for which $kx^2 + 4x + (k - 3) = 0$ has two distinct real roots.

1. two distinct real roots $\Rightarrow b^2 - 4ac > 0$: $16 - 4k(k - 3) > 0$ **M1**
2. $16 - 4k^2 + 12k > 0 \Rightarrow k^2 - 3k - 4 < 0$ (divided by -4, so the sign flips) **A1**
3. $(k - 4)(k + 1) < 0 \Rightarrow$ critical values -1 and 4 **M1**
4. **a U-shaped graph in k is negative between its roots: $-1 < k < 4$, $k \neq 0$** **A1**

- **Why k ≠ 0?** When k = 0 the equation becomes 4x − 3 = 0, which is linear and has only one root. Examiners reward students who notice this.

EXAM TIP

When you divide or multiply an inequality by a **negative** number, reverse the sign. Safer: move everything so the k^2 term is positive before factorising.

3 Solving quadratic equations

- **Factorise** when the numbers are friendly: check ac and look for two numbers that multiply to ac and add to b.
- **Formula** otherwise FORMULA LIST (MF19): $x = [-b \pm \sqrt{(b^2 - 4ac)}]/2a$. Write the substitution down, e.g. $x = [5 \pm \sqrt{(25 + 24)}]/6$.
- **Completing the square** when you need exact surd answers or the vertex anyway.

COMMON MISTAKE

Only writing the answers from a calculator's equation solver. Cambridge's rubric: "no marks will be given for unsupported answers from a calculator". Show the factorisation or the formula with numbers in.

4 Quadratic inequalities

The method

- **1 · Rearrange** so one side is 0 and the x^2 coefficient is positive.
- **2 · Critical values:** solve the equation = 0.
- **3 · Sketch** a U-shaped parabola through the critical values.
- **4 · Read off:** "> 0" means where the graph is **above** the axis (outside the roots); "< 0" means **below** (between the roots).

WORKED EXAMPLE

Solve $3x^2 - 5x - 2 > 0$.

1. $(3x + 1)(x - 2) = 0 \Rightarrow$ critical values $x = -\frac{1}{3}$ and $x = 2$ M1
2. the U-shaped graph is above the x-axis outside the roots
3. **$x < -\frac{1}{3}$ or $x > 2$** A1

WORKED EXAMPLE

Solve $x(x + 4) \leq 12$.

1. $x^2 + 4x - 12 \leq 0 \Rightarrow (x + 6)(x - 2) \leq 0$ M1
2. critical values −6 and 2; the graph is below or on the axis between them
3. **$-6 \leq x \leq 2$** A1

- **Writing the answer:** "$x < -\frac{1}{3}$ or $x > 2$" is two separate regions. "$-6 \leq x \leq 2$" is one region. Writing "$2 < x < -\frac{1}{3}$" is wrong, because no number satisfies it.

COMMON MISTAKE

Using ≤ where the question has <. Strict and non-strict inequalities must match the question; the June 2024 report flagged confusion between ≤ and <.

5 Where a line meets a curve

Substitute, then use the discriminant

- **Rearrange the linear equation** for one variable and substitute into the quadratic. You get a quadratic in one variable.
- **Its roots are the x-coordinates of the intersection points.** Substitute back into the **linear** equation to get y (it is simpler and avoids extra roots).

WORKED EXAMPLE

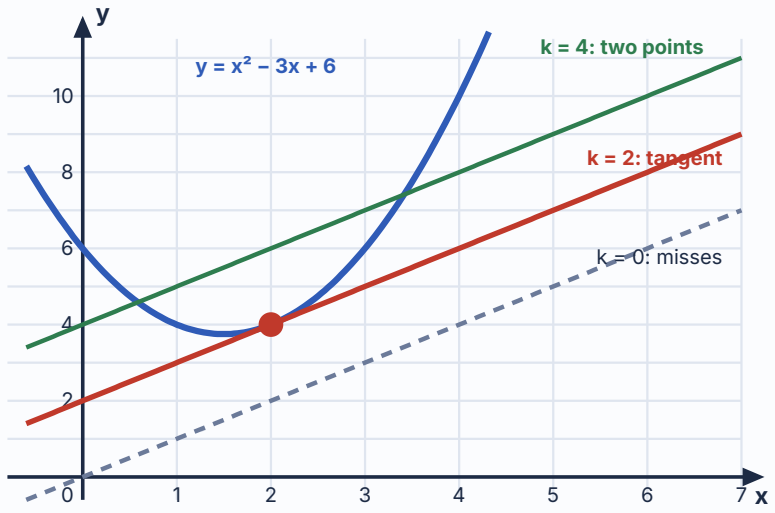
Find the points where $y = 3 - x$ meets $y = x^2 - 2x - 3$.

1. $x^2 - 2x - 3 = 3 - x \Rightarrow x^2 - x - 6 = 0$ M1
2. $(x - 3)(x + 2) = 0 \Rightarrow x = 3$ or $x = -2$ A1
3. **$y = 3 - x$ gives (3, 0) and (−2, 5)** A1

WORKED EXAMPLE

The line $y = 2x + c$ is a tangent to $y = x^2 - 4x + 7$. Find c and the point of contact.

1. $x^2 - 4x + 7 = 2x + c \Rightarrow x^2 - 6x + (7 - c) = 0$ M1
2. tangent ⇒ one repeated root ⇒ $b^2 - 4ac = 0$: $36 - 4(7 - c) = 0$ M1
3. $8 + 4c = 0 \Rightarrow$ **$c = -2$** A1
4. **$x^2 - 6x + 9 = 0 \Rightarrow (x - 3)^2 = 0 \Rightarrow x = 3$, $y = 2(3) - 2 = 4$: point (3, 4)** A1



Substitute $y = x + k$: $x^2 - 4x + (6 - k) = 0$, discriminant $4k - 8$. Tangent when $k = 2$.

The line $y = x + k$ cuts the curve twice, touches it once, or misses it, depending on the discriminant.

Discriminant of the combined quadratic	The line…
> 0	meets the curve at two points
= 0	is a tangent (touches at one point)
< 0	does not meet the curve

6 Quadratics in disguise

Spot the hidden quadratic

- **Look for a term that is the square of another:** $x^4 = (x^2)^2$, $x = (\sqrt{x})^2$, $x^{2/3} = (x^{1/3})^2$, $\tan^2 x = (\tan x)^2$.
- **Substitute u** for the inner expression, solve the quadratic in u, then go back to x. **Check each u is allowed:** $\sqrt{x}$ can't be negative, x^2 can't be negative.

WORKED EXAMPLE

Solve $x^4 - 5x^2 + 4 = 0$.

1. let $u = x^2$: $u^2 - 5u + 4 = 0$ M1
2. $(u - 1)(u - 4) = 0 \Rightarrow u = 1$ or $u = 4$ A1
3. **$x^2 = 1$ or $x^2 = 4 \Rightarrow x = \pm 1, \pm 2$** A1

WORKED EXAMPLE

Solve $2x - 7\sqrt{x} + 3 = 0$.

1. let $u = \sqrt{x}$: $2u^2 - 7u + 3 = 0$ M1
2. $(2u - 1)(u - 3) = 0 \Rightarrow u = \frac{1}{2}$ or $u = 3$ A1
3. **$\sqrt{x} = \frac{1}{2} \Rightarrow x = \frac{1}{4}$; $\sqrt{x} = 3 \Rightarrow x = 9$. Both u values are positive, so $x = \frac{1}{4}$ or $x = 9$** A1

WORKED EXAMPLE

Solve $6\tan^2\theta - \tan\theta - 2 = 0$ for $0° \le \theta \le 180°$.

1. $(3\tan\theta - 2)(2\tan\theta + 1) = 0 \Rightarrow \tan\theta = \frac{2}{3}$ or $\tan\theta = -\frac{1}{2}$ M1 A1
2. $\tan\theta = \frac{2}{3} \Rightarrow \theta =$ **33.7°**
3. **$\tan\theta = -\frac{1}{2} \Rightarrow \theta = 180° - 26.6° = 153.4°$** A1 A1

COMMON MISTAKE

Keeping an impossible solution. If $u = \sqrt{x}$ gives $u = -3$, reject it: $\sqrt{x}$ is never negative. In 2024, Paper 1 candidates lost a mark for keeping an extra invalid solution from a quadratic.

LINKING QUESTION

Completing the square gives the range of a function (Functions and transformations); the discriminant decides whether a line is a tangent to a circle (Coordinate geometry); and disguised quadratics in sin x and cos x appear in Trigonometry.

Test yourself — cover the answers first

QUICK CHECK

1. What does the completed square $a(x - p)^2 + q$ tell you about $y = ax^2 + bx + c$?
2. State the condition for $ax^2 + bx + c = 0$ to have real roots.
3. Solve $x^2 - x - 12 < 0$.
4. Express $x^2 + 10x + 3$ in completed square form.
5. Which substitution turns $x^{2/3} - 3x^{1/3} + 2 = 0$ into a quadratic?

EXAM-STYLE

6. Find the values of m for which the line $y = mx - 1$ is a tangent to the curve $y = x^2 + 3$.
7. $f(x) = -x^2 + 6x - 4$. (a) Express f(x) as $a - (x - b)^2$. (b) Find the range of f. (c) Solve $f(x) > 1$.
8. Find the set of values of k for which $x^2 + kx + 9 > 0$ for all real x.

Show answers

1. vertex (p, q); axis x = p; min value q if a > 0, max if a < 0
2. $b^2 - 4ac \geq 0$
3. $-3 < x < 4$
4. $(x + 5)^2 - 22$
5. $u = x^{1/3}$; u = 1 or 2, so x = 1 or 8
6. $x^2 - mx + 4 = 0$; $m^2 - 16 = 0$; $m = \pm 4$
7. (a) $5 - (x - 3)^2$
   (b) $f(x) \leq 5$
   (c) $x^2 - 6x + 5 < 0 \Rightarrow 1 < x < 5$
8. $k^2 - 36 < 0 \Rightarrow -6 < k < 6$

Stuck on a question?

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