A LEVEL MATHS 9709 · PAPER 1 PURE MATHEMATICS 1 · 1.6 · ~8 MIN READ

Series: binomial expansion, AP and GP

The binomial expansion for positive integer powers, specific terms and terms independent of x, arithmetic and geometric progressions, sums to infinity, and problems that link the two.

Binomial expansion | ${}^nC_r$ & Pascal | Specific coefficients | Term independent of x | Arithmetic progressions | Geometric progressions | Sum to infinity | Linked AP & GP

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Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded

highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **$(a + b)^n$** has n + 1 terms. The term containing b^r is **${}^nC_r\, a^{n-r}\, b^r$**. Put brackets round every term before raising it to a power.
- **Arithmetic progression (AP):** add the same d each time. $u_n = a + (n - 1)d$ and $S_n = \frac{1}{2}n[2a + (n - 1)d]$.
- **Geometric progression (GP):** multiply by the same r each time. $u_n = ar^{n-1}$ and $S_n = a(1 - r^n)/(1 - r)$. If $|r| < 1$, the sum to infinity is $a/(1 - r)$.
- **Three numbers a, b, c** are in AP if $2b = a + c$, and in GP if $b^2 = ac$.

BY THE END YOU CAN

- Expand $(a + b)^n$ and find any specific term or coefficient, including in products.
- Find the term independent of x.
- Solve problems with APs and GPs, including sums between two terms and the smallest n for a sum.
- Use the condition for convergence and the sum to infinity.

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1 The binomial expansion

Where the coefficients come from

- **$(a + b)^n = (a + b)(a + b)\ldots(a + b)$.** To get the term in $a^{n-r}b^r$, you choose b from r of the n brackets. There are ${}^nC_r$ ways to do that.

$$(a + b)^n = a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + b^n$$ FORMULA LIST (MF19)

n a positive integer; ${}^nC_r = n!/(r!(n - r)!)$

- **Pascal's triangle** gives the same numbers for small n: row 5 is 1, 5, 10, 10, 5, 1.
- **Symmetry:** ${}^nC_r = {}^nC_{n-r}$, e.g. ${}^8C_3 = {}^8C_5 = 56$.

WORKED EXAMPLE

Find the first three terms, in ascending powers of x, of $(2 - x)^5$.

1. $2^5 + {}^5C_1(2^4)(-x) + {}^5C_2(2^3)(-x)^2$ M1
2. **$= 32 - 80x + 80x^2$** A1 A1

COMMON MISTAKE

Losing brackets. $(3x)^2$ is $9x^2$, not $3x^2$. $(-x)^3$ is $-x^3$. The June 2024 report flagged expanding $(ax)^3$ as ax^3.

Specific terms and products

WORKED EXAMPLE

Find the coefficient of x^2 in the expansion of $(1 + 3x)(2 - x)^5$.

1. from $(2 - x)^5$: $32 - 80x + 80x^2 + \ldots$ B1
2. x^2 terms come from $1 \times 80x^2$ and $3x \times (-80x)$ M1
3. **coefficient $= 80 - 240 = -160$** A1

WORKED EXAMPLE

Find the term independent of x in the expansion of $(x^2 + 2/x)^6$.

1. general term ${}^6C_r(x^2)^{6-r}(2/x)^r = {}^6C_r\, 2^r\, x^{12 - 3r}$ M1
2. independent of x $\Rightarrow 12 - 3r = 0 \Rightarrow r = 4$ M1
3. **${}^6C_4 \times 2^4 = 15 \times 16 = 240$** A1

COMMON MISTAKE

Multiplying the wrong coefficients. In a product like $(1 + 3x)(2 - x)^5$, pair every term from the first bracket with the term that makes the right power. Students lost marks in 2024 by multiplying only one pair.

2 Arithmetic progressions

The formulas

$$u_n = a + (n-1)d$$ FORMULA LIST (MF19)

a = first term, d = common difference

$$S_n = \tfrac{1}{2}n[2a + (n-1)d] = \tfrac{1}{2}n(a + l)$$ FORMULA LIST (MF19)

l = last term

- **Why ½n(a + l)?** Write the sum forwards and backwards and add: every pair adds to a + l, and there are n pairs, so $2S_n = n(a + l)$. This is Gauss's trick.
- **Sum of the terms from the 11th to the 40th** = $S_{40} - S_{10}$, not $S_{40} - S_{11}$.

WORKED EXAMPLE

The 3rd term of an AP is 11 and the 10th term is 32. Find the sum of the first 20 terms.

1. $a + 2d = 11$, $a + 9d = 32$ M1
2. $7d = 21 \Rightarrow d = 3$, $a = 5$ A1
3. **$S_{20} = 10[10 + 19 \times 3] = 670$** M1 A1

WORKED EXAMPLE

An AP has first term 5 and common difference 3. Find the least number of terms for the sum to exceed 1000.

1. $\tfrac{1}{2}n[10 + 3(n-1)] > 1000 \Rightarrow 3n^2 + 7n - 2000 > 0$ M1
2. roots: $n = (-7 \pm \sqrt{24049})/6 \Rightarrow$ positive root $n = 24.68$ M1
3. n must be a whole number above 24.68: **n = 25** A1
4. **check: $S_{24} = 948$ and $S_{25} = 1025$ ✓**

COMMON MISTAKE

Rounding n to the nearest integer. 24.68 means 24 terms are not enough, so n = 25. Always check the two whole numbers on either side.

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3 Geometric progressions

The formulas

$$u_n = ar^{n-1}$$ FORMULA LIST (MF19)

r = common ratio

$$S_n = a(1 - r^n)/(1 - r)$$ FORMULA LIST (MF19)

use the form $a(r^n - 1)/(r - 1)$ when $r > 1$

$$S_\infty = a/(1 - r), \text{ for } |r| < 1$$ FORMULA LIST (MF19)

the condition for convergence

- **Why the sum formula works:** $S_n = a + ar + \ldots + ar^{n-1}$ and $rS_n = ar + \ldots + ar^n$. Subtract: $S_n - rS_n = a - ar^n$.
- **Why $|r| < 1$?** Then $r^n \to 0$ as n grows, so $S_n \to a/(1 - r)$. If $|r| \geq 1$, the terms don't shrink and the sum has no limit.

WORKED EXAMPLE

A GP has first term 24 and common ratio −½. Find the sum of the first 5 terms and the sum to infinity.

1. $S_5 = 24(1 - (-½)^5)/(1 - (-½)) = 24 \times (33/32)/(3/2) =$ **16.5** M1 A1
2. **$S_\infty = 24/(1 + ½) = 16$** M1 A1

WORKED EXAMPLE

The second term of a GP is 12 and its sum to infinity is 64. Find the possible values of r.

1. $ar = 12$ and $a/(1 - r) = 64$ B1
2. $a = 12/r \Rightarrow 12/(r(1 - r)) = 64 \Rightarrow 16r^2 - 16r + 3 = 0$ M1
3. **$(4r - 1)(4r - 3) = 0 \Rightarrow r = ¼$ or $r = ¾$** A1 **(both satisfy $|r| < 1$)**

EXAM TIP

"Show that the GP is convergent" needs the reason: **$|r| < 1$**, with the value of r shown.

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4 Linking an AP and a GP

Use $2b = a + c$ and $b^2 = ac$

WORKED EXAMPLE

The first, second and fourth terms of an AP (with $d \neq 0$) are the first three terms of a GP. Show that $a = d$, and find the common ratio of the GP.

1. AP terms: a, $a + d$, $a + 3d$ **B1**
2. in GP: $(a + d)^2 = a(a + 3d)$ **M1**
3. $a^2 + 2ad + d^2 = a^2 + 3ad \Rightarrow d^2 = ad \Rightarrow d = a$ (since $d \neq 0$) **A1**
4. **ratio $= (a + d)/a = 2a/a = 2$** **A1**

COMMON MISTAKE

Mixing up the letters. Use different letters for the AP and the GP (for example a, d and A, r), or say clearly which is which. The 2024 report noted confusion between p and d.

LINKING QUESTION

The binomial expansion extends to negative and fractional powers in Algebra (Paper 3), and binomial coefficients reappear in Permutations and combinations and the binomial distribution.

Test yourself — cover the answers first

QUICK CHECK

1. Write down ${}^7C_2$.
2. Find the coefficient of x^3 in $(1 + x/2)^8$.
3. An AP has $a = 4$, $d = -2$. Find u_{10}.
4. For which values of r does a GP have a sum to infinity?
5. Find S_∞ for $9 + 3 + 1 + \ldots$.

EXAM-STYLE

6. (a) Find the first three terms in the expansion of $(1 + ax)^6$ in ascending powers of x. (b) The coefficient of x^2 in $(2 - x)(1 + ax)^6$ is 36. Find the possible values of a.
7. An AP has sum of first 10 terms 175 and sum of first 20 terms 650. Find the first term and the common difference.
8. A ball is dropped from 2 m and each bounce reaches 80% of the previous height. Find the total distance travelled before it comes to rest.

Show answers

1. 21
2. ${}^{8}C_{3} \times (½)^{3} = 7$
3. −14
4. $-1 < r < 1$
5. 13.5
6. (a) $1 + 6ax + 15a^{2}x^{2}$
   (b) $30a^{2} - 6a = 36 \Rightarrow 5a^{2} - a - 6 = 0 \Rightarrow a = 6/5$ or $a = -1$
7. $5(2a + 9d) = 175$, $10(2a + 19d) = 650 \Rightarrow 2a + 9d = 35$, $2a + 19d = 65 \Rightarrow d = 3$, $a = 4$
8. $2 + 2 \times (1.6/(1 - 0.8)) = 2 + 16 = 18$ m

Stuck on a question?

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