A LEVEL MATHS 9709 · PAPER 1 PURE MATHEMATICS 1 · 1.5 · ~10 MIN READ

Trigonometry: graphs, identities and equations

Graphs of sine, cosine and tangent in degrees and radians, exact values, principal values of $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, the two key identities, and how to find every solution of a trig equation in an interval.

Trig graphs & transformations | Exact values | CAST & related angles | Principal values | $\tan\theta = \sin\theta/\cos\theta$ | $\sin^2\theta + \cos^2\theta = 1$ | Proving identities | Solving equations

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Paper tag = **where it is examined** FORMULA LIST (MF19) given in the exam M1 A1 B1 how marks are awarded

highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

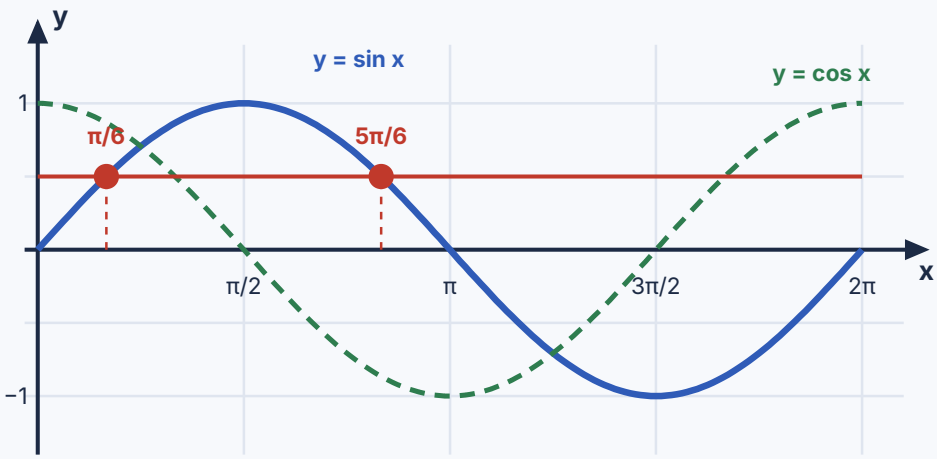
- **sin and cos repeat every 360° (2π); tan repeats every 180° (π).** Transformations change the amplitude, period and position exactly as in Functions and transformations.
- **Exact values** at 30°, 45° and 60° come from two triangles. Learn them; they are not in the formula list.
- **$\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ give one value only,** the principal value. Every other solution comes from the graph or the CAST diagram.
- **Two identities do almost all of Paper 1:** $\tan\theta \equiv \sin\theta/\cos\theta$ and $\sin^2\theta + \cos^2\theta \equiv 1$.
- **To solve an equation,** reduce it to one trig function, find the principal value, then list every solution in the interval. Never divide by sin x or cos x: factorise.

BY THE END YOU CAN

- Sketch and use the graphs of sin, cos and tan, including transformed graphs.
- Recall exact values and use related angles in all four quadrants.
- Use principal values of the inverse trig functions.
- Prove identities and solve equations using $\tan\theta \equiv \sin\theta/\cos\theta$ and $\sin^2\theta + \cos^2\theta \equiv 1$.
- Find all solutions in a given interval, in degrees or radians, including multiple angles like 2x.

1 Graphs

sin, cos and tan



sin x = ½ for 0 ≤ x ≤ 2π: x = π/6 and π − π/6 = 5π/6. Both lie on the line y = ½.

sin x and cos x for 0 ≤ x ≤ 2π. The line y = ½ meets y = sin x twice in one period.

Function	Period	Range	Key features
y = sin x	360° or 2π	−1 ≤ y ≤ 1	zeros at 0, π, 2π; max at π/2
y = cos x	360° or 2π	−1 ≤ y ≤ 1	max at 0 and 2π; zeros at π/2, 3π/2
y = tan x	180° or π	all real values	asymptotes at x = 90° (π/2), 270° (3π/2); zeros at 0, π

- **y = a sin(bx) + c:** amplitude |a|, period 360°/b (or 2π/b), centre line y = c. Range from c − |a| to c + |a|.
- **Example:** y = 1 − cos 2x has period π and range 0 ≤ y ≤ 2. It is y = cos x stretched by ½ parallel to the x-axis, reflected in the x-axis, then translated by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

WORKED EXAMPLE

The function f(x) = a + b cos x has maximum value 7 and minimum value −1. Find a and b, given b > 0.

1. maximum = a + b = 7, minimum = a − b = −1 M1
2. **a = 3, b = 4** A1 A1

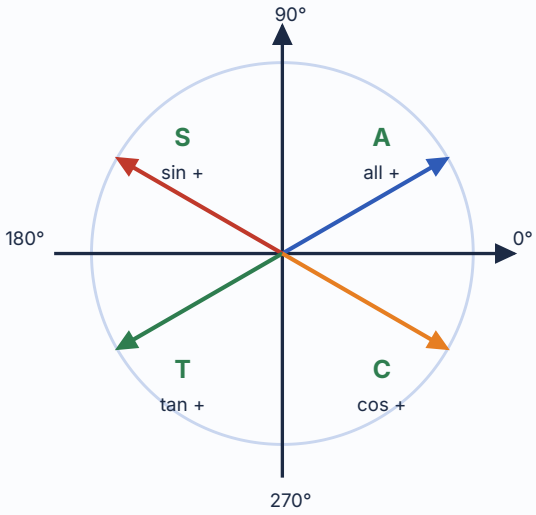
Riyaz Sir

2 Exact values and related angles

The two triangles

- **Half an equilateral triangle** (sides 1, 2, √3) gives 30° and 60°. **A right-angled isosceles triangle** (sides 1, 1, √2) gives 45°.

θ	0	30° = π/6	45° = π/4	60° = π/3	90° = π/2
sin θ	0	½	1/√2 = √2/2	√3/2	1
cos θ	1	√3/2	1/√2 = √2/2	½	0
tan θ	0	1/√3 = √3/3	1	√3	undefined



Related angles to α:
α (1st quadrant)
180° − α (2nd)
180° + α (3rd)
360° − α (4th)
Same basic angle, sign from CAST.

Angles measured anticlockwise from the positive x-axis. Example shown: α = 30°.

CAST: which functions are positive in each quadrant. Related angles share the same basic angle.

- **Related angles:** sin(180° − θ) = sin θ; cos(180° − θ) = −cos θ; tan(180° + θ) = tan θ; cos(360° − θ) = cos θ; sin(−θ) = −sin θ; cos(−θ) = cos θ.
- **Example:** sin 240° = −sin 60° = −√3/2 (3rd quadrant: only tan is positive).

Principal values

Inverse function	Principal value range
$\sin^{-1}x$	$-90° \le \sin^{-1}x \le 90°$ $(-\pi/2 \text{ to } \pi/2)$
$\cos^{-1}x$	$0° \le \cos^{-1}x \le 180°$ $(0 \text{ to } \pi)$
$\tan^{-1}x$	$-90° < \tan^{-1}x < 90°$ $(-\pi/2 \text{ to } \pi/2)$

- These ranges make each inverse a function: sin x is one-one on $-\pi/2 \le x \le \pi/2$, so it has an inverse there. They are in the formula list FORMULA LIST (MF19).
- **Example:** $\cos^{-1}(-½) = 2\pi/3$ (not $-\pi/3$, which is outside 0 to π). $\sin^{-1}(-½) = -\pi/6$.

3 Identities

The two you need

$$\tan\theta \equiv \sin\theta / \cos\theta$$ FORMULA LIST (MF19)

valid whenever $\cos\theta \neq 0$

$$\sin^2\theta + \cos^2\theta \equiv 1$$ FORMULA LIST (MF19)

Pythagoras on the unit circle; rearranges to $\sin^2\theta \equiv 1 - \cos^2\theta$, $\cos^2\theta \equiv 1 - \sin^2\theta$

- **Proving an identity:** start from one side (usually the more complicated one) and work to the other. Don't move terms across the ≡ sign.
- **Useful moves:** write tan as sin/cos; combine fractions over a common denominator; replace 1 with $\sin^2\theta + \cos^2\theta$; factorise.

WORKED EXAMPLE

Prove that $\tan\theta + 1/\tan\theta \equiv 1/(\sin\theta\cos\theta)$.

1. LHS = $\sin\theta/\cos\theta + \cos\theta/\sin\theta$ M1
2. = $(\sin^2\theta + \cos^2\theta)/(\sin\theta\cos\theta)$ M1
3. **= $1/(\sin\theta\cos\theta)$ = RHS** A1

WORKED EXAMPLE

Prove that $(1 - \cos^2\theta)/(1 - \sin^2\theta) \equiv \tan^2\theta$.

1. LHS = $\sin^2\theta/\cos^2\theta$ M1
2. **= $(\sin\theta/\cos\theta)^2 = \tan^2\theta$ = RHS** A1

COMMON MISTAKE

Working on both sides at once and meeting in the middle with no clear chain. Write one continuous chain: LHS = … = … = RHS.

4 Solving equations

The method

- **1 · Rearrange** to one trig function equal to a number: sin x = k, cos 2x = k or tan(x − 30°) = k.
- **2 · Adjust the interval** for the angle inside: if 0° ≤ x ≤ 180°, then 0° ≤ 2x ≤ 360°.
- **3 · Principal value** from the calculator, then **all solutions** in the adjusted interval using the graph or CAST.
- **4 · Undo** the inside (divide by 2, add 30°) and check each answer lies in the original interval.

WORKED EXAMPLE

Solve sin 2x = 0.6 for 0° ≤ x ≤ 180°.

1. 0° ≤ 2x ≤ 360° B1
2. 2x = 36.87° or 180° − 36.87° = 143.13° M1
3. **x = 18.4° or 71.6°** A1 A1

WORKED EXAMPLE

Solve tan(x − 30°) = −1 for 0° ≤ x ≤ 360°.

1. −30° ≤ x − 30° ≤ 330° B1
2. $\tan^{-1}(-1) = -45°$ (outside); add 180°: 135°, 315° M1
3. **x = 165° or 345°** A1 A1

COMMON MISTAKE

Rounding the inner angle first. If 2x = 36.869…°, keep all the digits and halve them; rounding 2x to 36.9° can change the final 1 d.p. This was flagged in the June 2024 report. Also note that the second solution of sin 2x = k is 180° − 2x, not 90° − x.

Equations that need an identity

- **Mixed sin and cos, one of them squared:** replace with $\sin^2x + \cos^2x = 1$ to get a quadratic in one function.
- **sin x and cos x to the same power:** divide by cos x to get tan x, but only after checking cos x = 0 isn't a solution.
- **A common factor:** factorise. Dividing by sin x throws away the solutions of sin x = 0.

WORKED EXAMPLE

Solve $2\sin^2x - \cos x - 1 = 0$ for $0 \le x \le 2\pi$.

1. $2(1 - \cos^2x) - \cos x - 1 = 0 \Rightarrow 2\cos^2x + \cos x - 1 = 0$ M1
2. $(2\cos x - 1)(\cos x + 1) = 0$ M1
3. $\cos x = \frac{1}{2} \Rightarrow x = \pi/3, 5\pi/3$; $\cos x = -1 \Rightarrow x = \pi$ A1 A1
4. **$x = \pi/3, \pi, 5\pi/3$**

WORKED EXAMPLE

Solve $3\sin^2 x = \sin x \cos x$ for $0° \le x \le 180°$.

1. $\sin x(3\sin x - \cos x) = 0$ M1
2. $\sin x = 0 \Rightarrow x = 0°, 180°$ A1
3. $3\sin x = \cos x \Rightarrow \tan x = \frac{1}{3} \Rightarrow x = 18.4°$ A1
4. **$x = 0°, 18.4°, 180°$**

WORKED EXAMPLE

Solve $2\sin x = 5\cos x$ for $0° \le x \le 360°$.

1. $\cos x = 0$ would force $\sin x = 0$ too, which is impossible, so dividing by $\cos x$ is safe: $\tan x = 2.5$ M1
2. **$x = 68.2°$ or $68.2° + 180° = 68.2°, 248.2°$** A1 A1

COMMON MISTAKE

Giving $\cos x = -\frac{1}{4}$ as an "answer". That is a step, not a solution: find the angles. And never keep a value like $\cos x = 3$, which has no solutions.

LINKING QUESTION

Quadratics in $\sin x$ or $\cos x$ use the methods of Quadratics; radian measure is in Circular measure; and Paper 3 extends these ideas to sec, cosec, cot and $R\sin(\theta + \alpha)$ in Further trigonometry.

Test yourself — cover the answers first

QUICK CHECK

1. Write down the exact value of $\cos 150°$.
2. What is the principal value of $\tan^{-1}(-\sqrt{3})$?
3. State the period of $y = \tan 3x$ in degrees.
4. If $0 \le x \le \pi$, what interval does $2x$ lie in?
5. Why must you not divide $2\sin x \cos x = \sin x$ by $\sin x$?

EXAM-STYLE

6. Solve $3\cos^2\theta + 8\sin\theta = 0$ for $0° \le \theta \le 360°$.
7. (a) Show that $3\sin\theta\tan\theta = 8$ can be written as $3\cos^2\theta + 8\cos\theta - 3 = 0$. (b) Hence solve $3\sin\theta\tan\theta = 8$ for $0° < \theta < 360°$.
8. The function f is defined by $f(x) = 4 - 3\sin 2x$ for $0 \le x \le \pi$. State the range of f and solve $f(x) = 5.5$.

Show answers

1. $-\sqrt{3}/2$
2. $-\pi/3$ $(-60°)$
3. $60°$
4. $0 \leq 2x \leq 2\pi$
5. you lose the solutions $\sin x = 0$
6. $3(1 - \sin^2\theta) + 8\sin\theta = 0 \Rightarrow 3s^2 - 8s - 3 = 0 \Rightarrow (3s + 1)(s - 3) = 0 \Rightarrow \sin\theta = -\frac{1}{3} \Rightarrow \theta = 199.5°, 340.5°$
7. (a) $3\sin^2\theta/\cos\theta = 8 \Rightarrow 3(1 - \cos^2\theta) = 8\cos\theta \Rightarrow 3\cos^2\theta + 8\cos\theta - 3 = 0$
   (b) $(3\cos\theta - 1)(\cos\theta + 3) = 0 \Rightarrow \cos\theta = \frac{1}{3}$ ($\cos\theta = -3$ impossible) $\Rightarrow \theta = 70.5°, 289.5°$
8. $1 \leq f(x) \leq 7$; $\sin 2x = -\frac{1}{2} \Rightarrow 2x = 7\pi/6, 11\pi/6 \Rightarrow x = 7\pi/12, 11\pi/12$

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