IB DP MATHS AA · TOPIC 1 NUMBER AND ALGEBRA · AHL 1.12, 1.13, 1.14 · ~6 MIN READ

Complex numbers

Cartesian form and arithmetic with complex numbers, the Argand diagram, modulus and argument, polar and Euler forms, conjugate roots of polynomials, De Moivre's theorem, nth roots and trig identities.

$i^2 = -1$ · Cartesian form · Argand diagram · Modulus & argument · Polar & Euler form · Conjugate roots · De Moivre · nth roots

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No tag = **SL and HL** · HL ONLY HL students only · FORMULA BOOKLET given in the exam

M1 A1 how marks are awarded · highlight = comes up again and again · Test yourself at the end

IN 30 SECONDS

- **HL only.** A complex number z = a + bi has real part a, imaginary part b, and $i^2 = -1$. Its **conjugate** is $z^* = a - bi$.
- On an **Argand diagram** z is a point; its distance from 0 is the **modulus** $|z| = r$ and its angle from the positive real axis is the **argument** θ.
- **Polar form** $z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta = re^{i\theta}$ makes multiplication easy: multiply the moduli, add the arguments.
- **De Moivre:** $(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}\,n\theta$. It gives powers, all n nth roots, and multiple-angle identities.
- Real polynomials have complex roots in **conjugate pairs**.

BY THE END YOU CAN

- Add, subtract, multiply and divide complex numbers in Cartesian form. HL
- Represent complex numbers on an Argand diagram and find modulus and argument. HL
- Convert between Cartesian, polar and Euler forms and multiply and divide in polar form. HL
- Use the conjugate root theorem to solve polynomial equations. HL
- Use De Moivre's theorem for powers, roots and trigonometric identities. HL

Riyaz Sir

1 Cartesian form HL

- **Arithmetic:** add real and imaginary parts separately; multiply out brackets using $i^2 = -1$; **divide by multiplying top and bottom by the conjugate** of the bottom.

WORKED EXAMPLE

Write $\frac{3 + 4i}{1 - 2i}$ in the form a + bi.

1. $\times \frac{1 + 2i}{1 + 2i}$: numerator $(3 + 4i)(1 + 2i) = 3 + 10i + 8i^2 = -5 + 10i$ M1 A1
2. denominator $1^2 + 2^2 = 5$
3. **$= -1 + 2i$** A1

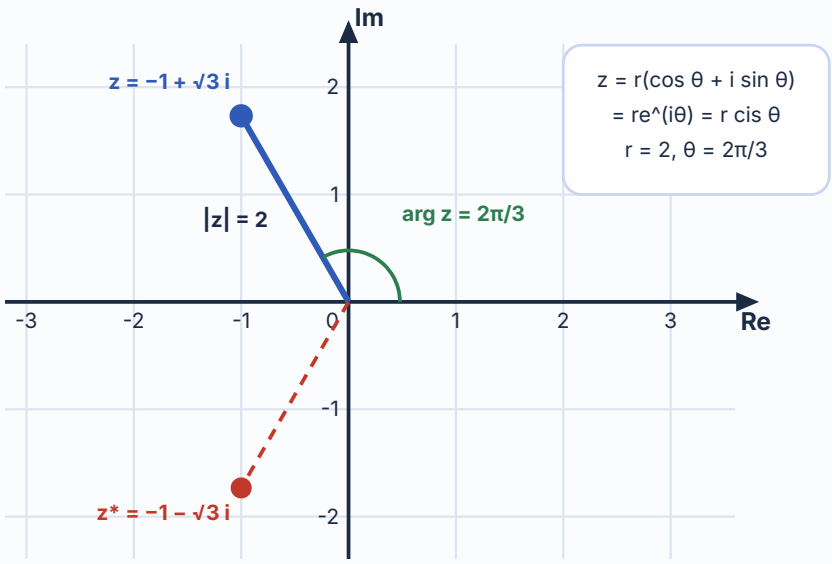
WORKED EXAMPLE

Solve $z^2 + 4z + 13 = 0$.

1. $z = \frac{-4 \pm \sqrt{(16 - 52)}}{2} = \frac{-4 \pm 6i}{2}$ M1
2. **$z = -2 \pm 3i$** A1

- **Equal complex numbers** have equal real parts AND equal imaginary parts: one complex equation gives two real equations.

2 Modulus, argument and the Argand diagram HL



Modulus is length, argument is angle. The conjugate is the reflection in the real axis.

$$|z| = r = \sqrt{(a^2 + b^2)}, \arg z = \theta \text{ with } \tan\theta = \frac{b}{a}$$

choose θ in the correct quadrant (usually $-\pi < \theta \leq \pi$)

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} = r\,\text{cis}\,\theta$$ FORMULA BOOKLET

polar and Euler forms

WORKED EXAMPLE

Write $z = -1 + \sqrt{3}\,i$ in polar form.

1. $r = \sqrt{(1 + 3)} = 2$ A1
2. z is in the 2nd quadrant: $\theta = \pi - \pi/3 =$ **$2\pi/3$** M1 A1
3. **$z = 2\,\text{cis}\,(2\pi/3) = 2e^{2\pi i/3}$** A1

COMMON MISTAKE

Taking $\theta = \arctan(b/a)$ without checking the quadrant. $\arctan(\sqrt{3}/-1)$ gives $-\pi/3$, which points into the 4th quadrant: wrong.

- **Multiplying:** $|z_1z_2| = |z_1||z_2|$ and $\arg(z_1z_2) = \arg z_1 + \arg z_2$. **Dividing:** divide moduli, subtract arguments. Geometrically, multiplying by cis θ rotates a point by θ.

3 Conjugate roots HL

- **Conjugate root theorem:** if a polynomial has **real** coefficients and $a + bi$ is a root, then $a - bi$ is also a root. Their product $(z - (a + bi))(z - (a - bi)) = z^2 - 2az + (a^2 + b^2)$ is a real quadratic factor.

WORKED EXAMPLE

Given that $3i$ is a root of $z^3 - 2z^2 + 9z - 18 = 0$, find the other roots.

1. $-3i$ is also a root, so $(z^2 + 9)$ is a factor M1
2. $z^3 - 2z^2 + 9z - 18 = (z^2 + 9)(z - 2)$ A1
3. **roots $3i$, $-3i$, 2** A1

4 De Moivre's theorem HL

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) = r^ne^{in\theta}$$ FORMULA BOOKLET

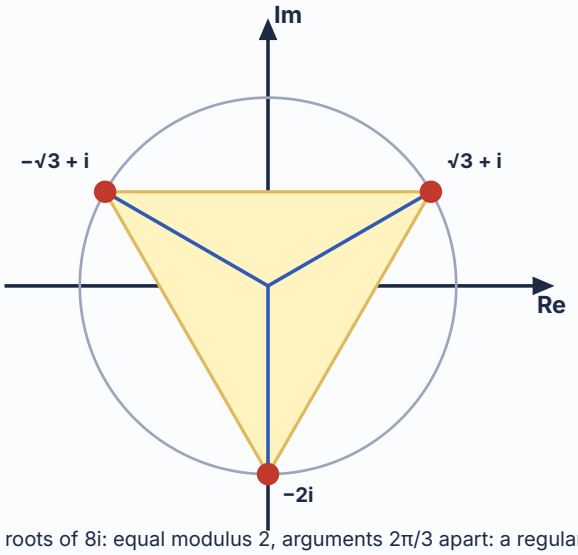
true for all $n \in \mathbb{Z}$ (and proved by induction for $n \in \mathbb{Z}^+$)

WORKED EXAMPLE

Find $(1 + i)^8$.

1. $1 + i = \sqrt{2}\,\text{cis}\,(\pi/4)$ A1
2. $(1 + i)^8 = (\sqrt{2})^8\,\text{cis}\,2\pi$ M1
3. **$= 16$** A1

nth roots



The 3 cube roots of 8i: equal modulus 2, arguments 2π/3 apart: a regular triangle.

The n roots of any complex number are equally spaced around a circle.

WORKED EXAMPLE

Solve $z^3 = 8i$.

1. $8i = 8\,\text{cis}(\pi/2 + 2k\pi)$ M1
2. $z = 2\,\text{cis}(\frac{\pi}{6} + \frac{2k\pi}{3})$, k = 0, 1, 2 M1 A1
3. **$z = \sqrt{3} + i, -\sqrt{3} + i, -2i$** A1

- **Roots of unity:** $z^n = 1$ has roots $\text{cis}(2k\pi/n)$, k = 0, 1, …, n − 1. They add up to zero.

Trig identities from De Moivre

WORKED EXAMPLE

Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

1. $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ (De Moivre) M1
2. expand: real part = $\cos^3\theta - 3\cos\theta\sin^2\theta$ A1
3. **$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) = 4\cos^3\theta - 3\cos\theta$** A1 AG

EXAM TIP

Euler's form makes powers and products quickest: $(re^{i\theta})^n = r^n e^{in\theta}$. Convert back to Cartesian only at the end if the question asks.

GDC (CALCULATOR) METHOD

Paper 2: set the GDC to complex mode (a + bi or $re^{i\theta}$) to check arithmetic, moduli and arguments. Still write the method: polar form, then De Moivre.

LINKING QUESTION

Complex numbers make rotations and oscillations simple: physics and engineering use $e^{i\omega t}$ for waves and AC circuits, and Euler's formula links e, i and π: $e^{i\pi} + 1 = 0$.

IDEA FOR YOUR IA / EXPLORATION

Explore fractals built from complex iteration (the Mandelbrot set, $z \to z^2 + c$), or the geometry of the nth roots of unity and regular polygons.

Test yourself — cover the answers first

QUICK CHECK

1. Simplify i^3 and i^4.
2. Find $|3 - 4i|$.

3. Find the conjugate of $2 - 5i$.
4. What happens to z on an Argand diagram when multiplied by i?
5. Write $4e^{i\pi/2}$ in Cartesian form.
6. How many roots does $z^5 = 1$ have?

EXAM-STYLE

7. Given $z = 2\text{cis}(\pi/3)$ and $w = 3\text{cis}(\pi/6)$, find zw and z/w in polar form.
8. Find all solutions of $z^4 = -16$.

Show answers

1. $-i$ and 1
2. 5
3. $2 + 5i$
4. it rotates by $\pi/2$ anticlockwise
5. $4i$
6. 5
7. $zw = 6\ \text{cis}(\pi/2) = \mathbf{6i}$
   $z/w = \frac{2}{3}\ \text{cis}(\pi/6)$
8. $-16 = 16\ \text{cis}(\pi + 2k\pi)$
   $z = 2\ \text{cis}(\pi/4 + k\pi/2)$
   $z = \sqrt{2} \pm \sqrt{2}\,i, -\sqrt{2} \pm \sqrt{2}\,i$

Stuck on a question?

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