IB DP MATHS AA · TOPIC 5 CALCULUS · AHL 5.18, 5.19 · ~9 MIN READ

Differential equations and Maclaurin series

HL first-order differential equations: separating variables, slope fields, homogeneous equations with y = vx, integrating factors and modelling growth and cooling, plus Maclaurin series from the formula booklet, building new series by substitution, multiplication and differentiation, series from a differential equation, and limits using series.

Differential equations | Separating variables | Slope fields | Homogeneous equations (y = vx) | Integrating factor | Exponential models | Newton's law of cooling | Maclaurin series

RS by Riyaz Sir
mentorriyaz.in

No tag = **SL and HL** HL ONLY HL students only FORMULA BOOKLET given in the exam

M1 A1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- A **differential equation** links y, x and dy/dx. **Solving** it means finding y as a function of x; a boundary condition fixes the constant.
- **Three methods:** separate the variables (dy/dx = f(x)g(y)); substitute y = vx for **homogeneous** equations (dy/dx = f(y/x)); multiply by an **integrating factor** $e^{\int P\,dx}$ for dy/dx + P(x)y = Q(x).
- A **Maclaurin series** writes f(x) as a polynomial in x: f(x) = f(0) + xf′(0) + $\frac{x^2}{2!}$f″(0) + … The series for e^x, sin x, cos x, ln(1 + x) and $(1 + x)^p$ are in the booklet.
- **Build new series** from known ones by substituting, multiplying, differentiating or integrating, instead of differentiating from scratch.
- This note is HL only (AHL 5.18–5.19).

BY THE END YOU CAN

- Set up differential equations from a description of rates of change. HL
- Solve separable differential equations with boundary conditions. HL
- Sketch and interpret slope fields. HL
- Solve homogeneous equations with y = vx, and linear equations with an integrating factor. HL
- Find Maclaurin series by differentiation and from the standard series. HL
- Use Maclaurin series to approximate values, solve DEs and find limits. HL

SYLLABUS NOTE

From first exams in 2029, Euler's method for numerical solutions of differential equations is removed. If you sit exams before 2029, you still need it: $y_{n+1} = y_n + h \times f(x_n, y_n)$, $x_{n+1} = x_n + h$.

1 Setting up and separating variables HL

- **Words to equations:** "the rate of growth is proportional to the population" means $dP/dt = kP$. "Cools at a rate proportional to the difference from room temperature" means $dT/dt = -k(T - T_{room})$.
- **Separating:** get all the y terms (with dy) on one side and all the x terms (with dx) on the other, then integrate both sides. One + C is enough.

WORKED EXAMPLE

Solve $\frac{dy}{dx} = xy$ given $y = 2$ when $x = 0$.

1. $\int \frac{1}{y}\,dy = \int x\,dx$ M1
2. $\ln|y| = \frac{x^2}{2} + C$ A1
3. $y = Ae^{x^{2}/2}$; $y(0) = 2 \Rightarrow A = 2$ M1
4. **$y = 2e^{x^{2}/2}$** A1

WORKED EXAMPLE

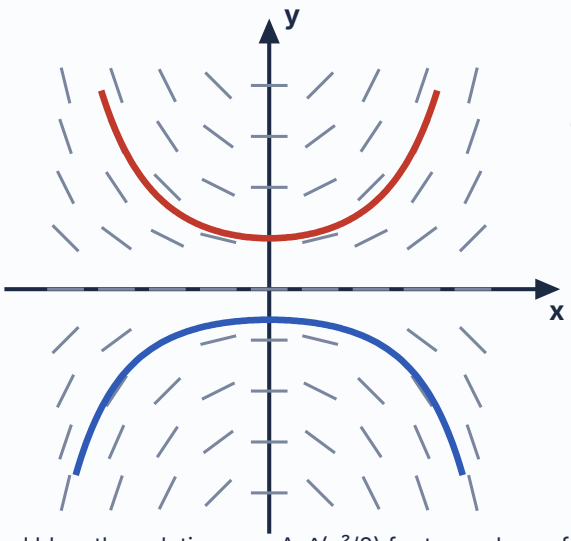
Solve $\frac{dy}{dx} = 1 + y^2$ with $y = 0$ at $x = 0$.

1. $\int \frac{1}{1+y^2}\,dy = \int 1\,dx$ M1
2. $\arctan y = x + C$, and $C = 0$ A1
3. **$y = \tan x$ (for $-\pi/2 < x < \pi/2$)** A1

COMMON MISTAKE

Writing $\ln y = \frac{x^2}{2} + C$ and then $y = e^{x^{2}/2} + C$. The constant goes inside the exponential: $y = e^{x^{2}/2 + C} = Ae^{x^{2}/2}$.

2 Slope fields HL



dy/dx = xy

each dash has gradient xy

Red and blue: the solutions y = Ae^(x²/2) for two values of A.

At each point, draw a short dash with the gradient given by the DE. Solution curves follow the dashes and never cross.

- **Reading a slope field:** where dy/dx = 0 the dashes are horizontal (here along both axes). Isoclines are curves where the gradient is the same.

3 Homogeneous equations HL

- **Recognise it:** $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, for example $\frac{x+y}{x} = 1 + \frac{y}{x}$ or $\frac{x^2+y^2}{xy}$.
- **Method:** let $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$. The equation becomes separable in v and x.

WORKED EXAMPLE

Solve $\frac{dy}{dx} = \frac{x+y}{x}$ for $x > 0$, given $y = 2$ when $x = 1$.

1. $y = vx$: $v + x\frac{dv}{dx} = 1 + v$ M1
2. $\frac{dv}{dx} = \frac{1}{x} \Rightarrow v = \ln x + C$ A1
3. $y = x \ln x + Cx$; $y(1) = 2 \Rightarrow C = 2$ M1
4. **$y = x \ln x + 2x$** A1

4 The integrating factor HL

$$\frac{dy}{dx} + P(x)y = Q(x), \text{ integrating factor } e^{\int P(x)\,dx}$$ FORMULA BOOKLET

multiply through and the left side becomes d/dx (IF × y)

WORKED EXAMPLE

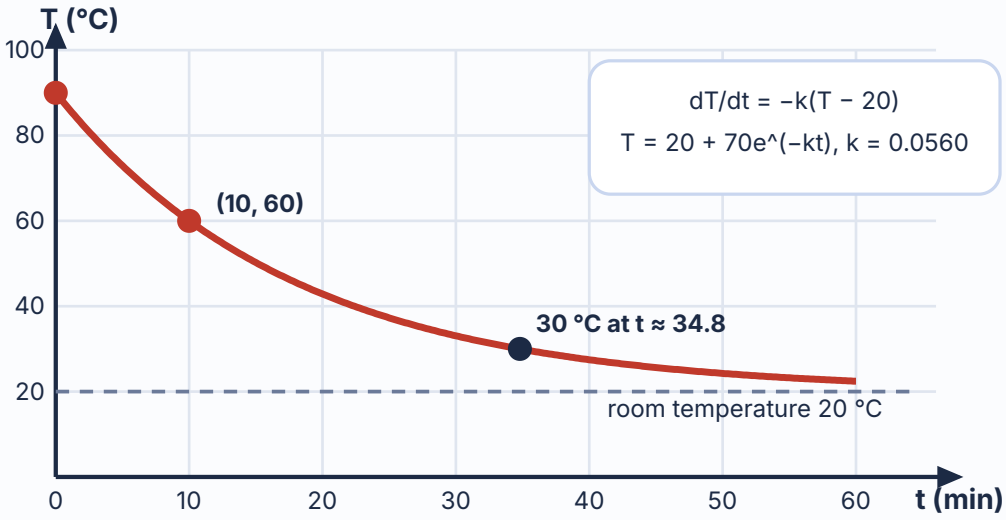
Solve $\frac{dy}{dx} + \frac{2y}{x} = x^2$ for $x > 0$, given $y = 1$ when $x = 1$.

1. IF $= e^{\int 2/x\,dx} = e^{2\ln x} = x^2$ M1 A1
2. $\frac{d}{dx}(x^2y) = x^4 \Rightarrow x^2y = \frac{x^5}{5} + C$ M1
3. $y(1) = 1 \Rightarrow C = \frac{4}{5}$, so $y = \frac{x^3}{5} + \frac{4}{5x^2}$ A1

COMMON MISTAKE

Forgetting to put the DE in standard form first. dy/dx must have coefficient 1: divide $x\frac{dy}{dx} + 2y = x^3$ by x before finding P(x).

5 Models HL



The temperature drops quickly at first, then levels off towards room temperature.

WORKED EXAMPLE

Tea at 90 °C cools in a 20 °C room, with $\frac{dT}{dt} = -k(T - 20)$. After 10 minutes it is 60 °C. When does it reach 30 °C?

1. separate: $T = 20 + 70e^{-kt}$ M1 A1
2. $60 = 20 + 70e^{-10k} \Rightarrow k = \frac{\ln(7/4)}{10} = 0.0560$ M1 A1
3. **$30 = 20 + 70e^{-kt} \Rightarrow t = \frac{\ln 7}{k} = 34.8$ min** A1

- **Logistic growth** $dP/dt = kP(1 - P/N)$ is solved with partial fractions (Proof, counting and partial fractions); the population levels off at N.

GDC (CALCULATOR) METHOD

Paper 2: graph your solution and check it passes through the boundary point. Some GDCs can draw slope fields, which helps you check the shape.

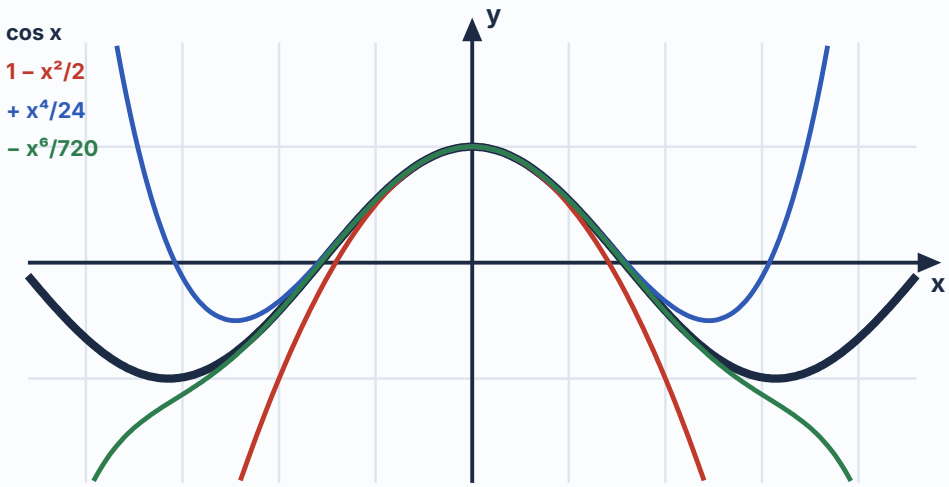
6 Maclaurin series HL

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$ FORMULA BOOKLET

general Maclaurin series

Function	Series	Valid for
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	all x
$\ln(1 + x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$	$-1 < x \leq 1$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	all x
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$	all x
$\arctan x$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$	$\lvert x\rvert \leq 1$
$(1 + x)^p$	$1 + px + \frac{p(p-1)}{2!}x^2 + \ldots$	$\lvert x\rvert < 1$

Riyaz Sir



More terms: a good fit over a wider interval around x = 0.

Each extra term of the cos x series hugs the curve over a wider interval.

WORKED EXAMPLE

Find the Maclaurin series for e^{2x} up to the x^3 term.

1. substitute 2x for x in the e^x series **M1**
2. $1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6}$ **M1**
3. **$= 1 + 2x + 2x^2 + \frac{4}{3}x^3$** **A1**

WORKED EXAMPLE

Find the series for $e^x \sin x$ up to the x^3 term.

1. $(1 + x + \frac{x^2}{2} + \ldots)(x - \frac{x^3}{6} + \ldots)$ **M1**
2. $= x + x^2 + \frac{x^3}{2} - \frac{x^3}{6} + \ldots$ **A1**
3. **$= x + x^2 + \frac{x^3}{3}$** **A1**

- **Differentiate or integrate a series:** differentiating sin x's series gives cos x's; integrating $\frac{1}{1+x^2} = 1 - x^2 + x^4 - \ldots$ term by term gives arctan x.

WORKED EXAMPLE

Use the series for $e^{-x^{2}}$ to estimate $\int_0^{0.5} e^{-x^{2}}\,dx$.

1. $e^{-x^{2}} = 1 - x^2 + \frac{x^4}{2} - \ldots$ **M1**
2. integrate: $x - \frac{x^3}{3} + \frac{x^5}{10}$ from 0 to 0.5 **M1**
3. **$= 0.5 - 0.04167 + 0.00313 = 0.4615$ (true value 0.4613)** **A1**

7 Series from a differential equation HL

WORKED EXAMPLE

y satisfies $\frac{dy}{dx} = x + y$ with $y(0) = 1$. Find the Maclaurin series up to the x^3 term.

1. $y'(0) = 0 + 1 = 1$ A1
2. differentiate: $y'' = 1 + y' \Rightarrow y''(0) = 2$; $y''' = y'' \Rightarrow y'''(0) = 2$ M1 A1
3. **$y \approx 1 + x + x^2 + \frac{x^3}{3}$** A1

8 Limits with series HL

WORKED EXAMPLE

Find $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

1. $1 - \cos x = \frac{x^2}{2} - \frac{x^4}{24} + \ldots$ M1
2. $\frac{1 - \cos x}{x^2} = \frac{1}{2} - \frac{x^2}{24} + \ldots \to \frac{1}{2}$ A1

EXAM TIP

For a 0/0 limit, series and L'Hôpital's rule (Differentiation) both work. Series are often quicker when the function is a product or a composite.

LINKING QUESTION

Every DE here is solved by integrating (Integration and kinematics). Exponential models link to Exponents and logarithms, and the $(1 + x)^p$ series extends the binomial theorem (Sequences, series and the binomial theorem) to any power p.

IDEA FOR YOUR IA / EXPLORATION

Model a real cooling curve (a cup of tea with a thermometer and timer) with Newton's law of cooling and compare the fitted k with your data, or investigate how many Maclaurin terms you need to compute π or e to a given accuracy.

Test yourself — cover the answers first

QUICK CHECK

1. HL Write "the rate of decay of mass m is proportional to m" as a DE.
2. HL What substitution solves a homogeneous DE?
3. HL Find the integrating factor for $dy/dx + 3y = e^x$.
4. HL Write the first three non-zero terms of the series for $\sin 2x$.
5. HL What is $f''(0)$ if the Maclaurin series of f starts $3 + 2x + 5x^2 + \ldots$?

EXAM-STYLE

6. HL Use an integrating factor to solve $x\frac{dy}{dx} - y = x^2$, $x > 0$, given $y = 3$ at $x = 1$.

7. HL Find the Maclaurin series of $\ln(1 + \sin x)$ up to the x^2 term.

Show answers

1. $dm/dt = -km$
2. $y = vx$
3. e^{3x}
4. $2x - (4/3)x^3 + (4/15)x^5$
5. 10
6. standard form $\frac{dy}{dx} - \frac{y}{x} = x$; IF $= e^{-\ln x} = \frac{1}{x}$

   $\frac{d}{dx}\left(\frac{y}{x}\right) = 1 \Rightarrow \frac{y}{x} = x + C$

   $y(1) = 3 \Rightarrow C = 2$, so $\mathbf{y = x^2 + 2x}$
7. $\sin x \approx x$, so $\ln(1 + \sin x) \approx \sin x - \frac{\sin^2 x}{2}$

   $\approx (x - \ldots) - \frac{x^2}{2}$

   $\mathbf{\approx x - \frac{x^2}{2}}$

Stuck on a question?

Send it to me on WhatsApp. I read every message.

Want regular help? Send this to your parent →

Share with a friend

Ask Riyaz Sir