IB DP MATHS AA · TOPIC 1 NUMBER AND ALGEBRA · SL 1.5, 1.7 · SL 2.9 · ~7 MIN READ

Exponents and logarithms

Laws of exponents, logarithms as inverses of exponentials, natural logs and e, the laws of logarithms and change of base, solving exponential and log equations, and exponential and logarithmic functions and models.

Laws of exponents | Logarithms | Natural log & e | Laws of logs | Change of base | Exponential equations | Exponential & log graphs | Growth & decay models

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No tag = **SL and HL** HL ONLY HL students only FORMULA BOOKLET given in the exam

M1 A1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- A **logarithm** answers "what power?": $a^x = b \Leftrightarrow x = \log_a b$. Logs undo exponentials, and the other way round.
- The **laws of logs** turn products into sums, quotients into differences and powers into multipliers. Use them to bring an unknown down from an exponent.
- **e** ≈ 2.718 is the natural base; $\ln x = \log_e x$. Exponential models have the form $y = Ae^{kt}$ (growth if $k > 0$, decay if $k < 0$).
- The graphs of $y = a^x$ and $y = \log_a x$ are reflections in $y = x$, with asymptotes $y = 0$ and $x = 0$.
- This note is the same at SL and HL, and unchanged for 2029.

BY THE END YOU CAN

- Use the laws of exponents with integer and rational powers.
- Convert between exponential and log form and use the laws of logarithms and change of base.
- Solve exponential and logarithmic equations, analytically and with a GDC.
- Sketch and interpret exponential and log graphs, and use exponential models.

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1 Laws of exponents

Law	Example
$a^m \times a^n = a^{m+n}$	$2^3 \times 2^4 = 2^7$
$a^m \div a^n = a^{m-n}$	$x^5/x^2 = x^3$
$(a^m)^n = a^{mn}$	$(3^2)^4 = 3^8$
$a^0 = 1$, $a^{-n} = 1/a^n$	$5^{-2} = 1/25$
$a^{m/n} = (\sqrt[n]{a})^m$	$8^{2/3} = 2^2 = 4$

WORKED EXAMPLE

Simplify $(27x^6)^{-2/3}$.

1. $= 27^{-2/3} \times x^{-4}$ M1
2. $= \frac{1}{9x^4}$ A1

- **Same base trick:** to solve $4^{x+1} = 8^x$, write both as powers of 2: $2^{2x+2} = 2^{3x}$, so $2x + 2 = 3x$ and $x = 2$.

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2 Logarithms

$$a^x = b \Leftrightarrow x = \log_a b$$ FORMULA BOOKLET

$a > 0, b > 0, a \neq 1$

- **Special values:** $\log_a 1 = 0$, $\log_a a = 1$, $\log_a a^x = x$, $a^{\log_a x} = x$. log x means $\log_{10} x$; ln x means $\log_e x$.

Law of logarithms FORMULA BOOKLET	In words
$\log_a xy = \log_a x + \log_a y$	product → sum
$\log_a (x/y) = \log_a x - \log_a y$	quotient → difference
$\log_a x^m = m \log_a x$	power → multiplier

$$\log_b a = \frac{\log_c a}{\log_c b}$$ FORMULA BOOKLET

change of base

WORKED EXAMPLE

Write 2 ln 3 − ln 6 + ln 4 as a single logarithm.

1. $= \ln 9 - \ln 6 + \ln 4$ M1
2. $= \ln \frac{9 \times 4}{6} = \ln 6$ A1

WORKED EXAMPLE

Evaluate $\log_8 32$ without a calculator.

1. $\log_8 32 = \frac{\log_2 32}{\log_2 8}$ M1
2. $= \frac{5}{3}$ A1

COMMON MISTAKE

Writing log(x + y) = log x + log y. There is no law for the log of a sum. The product law is log(xy).

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3 Solving equations

WORKED EXAMPLE

Solve $3^{2x+1} = 20$, giving x to 3 s.f.

1. take ln of both sides: $(2x + 1) \ln 3 = \ln 20$ M1
2. $2x + 1 = \ln 20 / \ln 3 = 2.727$ A1
3. **$x = 0.863$** A1

WORKED EXAMPLE

Solve $\log_2 x + \log_2(x - 2) = 3$.

1. $\log_2 x(x - 2) = 3$, so $x(x - 2) = 2^3 = 8$ M1
2. $x^2 - 2x - 8 = 0 \Rightarrow x = 4$ or $x = -2$ A1
3. **$\log_2(-2)$ is undefined, so $x = 4$ only** R1

WORKED EXAMPLE

Solve $e^{2x} - 5e^x + 6 = 0$.

1. let $u = e^x$: $u^2 - 5u + 6 = 0$ M1
2. $u = 2$ or 3 A1
3. **$x = \ln 2$ or $x = \ln 3$** A1

EXAM TIP

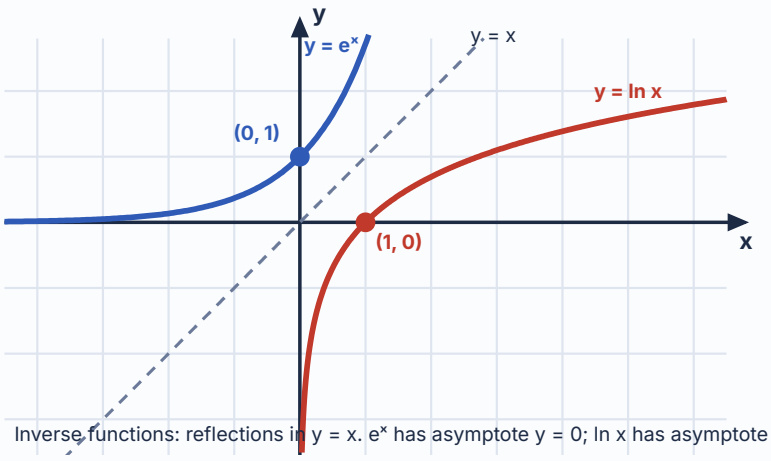
Paper 1 log equations almost always have a "reject" step: check every solution in the ORIGINAL equation, because logs of negative numbers and zero are undefined.

GDC (CALCULATOR) METHOD

Paper 2: graph both sides ($y = 3^{2x+1}$ and $y = 20$) and use intersect, or use the numerical solver. Write the equation you solved and give the answer to 3 s.f.

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4 Exponential and logarithmic functions



Inverse functions: reflections in y = x. e^x has asymptote y = 0; ln x has asymptote x = 0.

e^x and ln x undo each other, so their graphs mirror in y = x.

	$y = a^x$ $(a > 1)$	$y = \log_a x$
domain	$x \in \mathbb{R}$	$x > 0$
range	$y > 0$	$y \in \mathbb{R}$
intercept	(0, 1)	(1, 0)
asymptote	$y = 0$	$x = 0$

- **e^x is special:** its gradient at every point equals its value, which is why it is the natural choice for growth models (see Differentiation).

$$a^x = e^{x \ln a}$$ FORMULA BOOKLET

any exponential can be written with base e

Exponential models

- **Growth:** $N = N_0e^{kt}$, $k > 0$. **Decay:** $N = N_0e^{-kt}$. Doubling time = ln 2/k; half-life = ln 2/k.

WORKED EXAMPLE

A population is modelled by $P = 500e^{0.04t}$ (t in years). Find the initial population and when it doubles.

1. t = 0: P = **500** A1
2. $1000 = 500e^{0.04t} \Rightarrow e^{0.04t} = 2$ M1
3. **t = ln 2/0.04 = 17.3 years** A1

WORKED EXAMPLE

A cup of tea cools as $T = 20 + 70e^{-0.05t}$ (°C, t in minutes). When is it 50 °C, and what is the long-term temperature?

1. $30 = 70e^{-0.05t} \Rightarrow t = -\ln(3/7)/0.05 =$ **16.9 min** M1 A1
2. **as $t \to \infty$, $e^{-0.05t} \to 0$, so $T \to 20$ °C (room temperature)** A1

LINKING QUESTION

Logs straighten exponential data: if $y = Ae^{kx}$ then $\ln y = \ln A + kx$, a straight line. Physics uses the same trick for radioactive decay (E.3).

IDEA FOR YOUR IA / EXPLORATION

Fit an exponential (or logistic) model to real data: COVID cases, cooling coffee, a bouncing ball's heights, and compare models using a log-transformed graph.

Test yourself — cover the answers first

QUICK CHECK

1. Write $\log_3 81 = 4$ in exponential form.
2. Simplify $\log_a a^5$.
3. Solve $2^x = 32$.
4. What is the domain of $y = \ln(x - 3)$?
5. Expand $\ln(x^2y/z)$.
6. Find the half-life for $N = N_0e^{-0.2t}$.

EXAM-STYLE

7. Solve $2\log_3 x - \log_3(x + 6) = 1$.
8. A car worth \$30 000 loses 15 % of its value each year. After how many whole years is it first worth less than \$10 000?
9. Show that $e^{x \ln 2} = 2^x$ and hence solve $2^x = e^3$.

Show answers

1. $3^4 = 81$
2. 5
3. $x = 5$
4. $x > 3$
5. $2\ln x + \ln y - \ln z$
6. $\ln 2/0.2 = 3.47$
7. $\log_3(x^2/(x + 6)) = 1$
   $x^2 = 3x + 18 \Rightarrow x = 6$ or $x = -3$
   $x = 6$ ($x = -3$ rejected)
8. $30\,000 \times 0.85^n < 10\,000$
   $n > \ln(1/3)/\ln 0.85 = 6.76$
   7 years
9. $e^{x \ln 2} = (e^{\ln 2})^x = 2^x$
   $x \ln 2 = 3$
   $x = 3/\ln 2 \approx 4.33$

Stuck on a question?

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