IB DP MATHS AA · TOPIC 5 CALCULUS · SL 5.5, 5.9–5.11 · AHL 5.15–5.17 · ~9 MIN READ

Integration and kinematics

Antiderivatives and the constant of integration, definite integrals and areas, area between curves, integrating e^x, 1/x, sin and cos, reverse chain rule, kinematics with displacement and distance, and for HL: substitution, integration by parts, arctan and arcsin integrals, and volumes of revolution.

Indefinite integrals · Boundary conditions · Definite integrals · Area under a curve · Area between curves · e^x, 1/x, sin, cos · Reverse chain rule · Kinematics

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No tag = **SL and HL** · HL ONLY HL students only · FORMULA BOOKLET given in the exam

M1 A1 how marks are awarded · highlight = comes up again and again · Test yourself at the end

IN 30 SECONDS

- **Integration undoes differentiation.** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$. Never forget + C on an indefinite integral.
- **Definite integral** $\int_a^b f(x)\,dx = F(b) - F(a)$ is the **signed** area: regions below the x-axis count as negative.
- **Area between curves** = ∫(top − bottom) dx between the intersection points.
- **Kinematics:** v = ds/dt and a = dv/dt, so s = ∫v dt and v = ∫a dt. **Displacement** = ∫v dt; **distance** = ∫|v| dt.
- **HL:** substitution, integration by parts, $\int \frac{1}{a^2 + x^2}$ and $\int \frac{1}{\sqrt{(a^2 - x^2)}}$, and volumes of revolution $V = \pi\int y^2\,dx$.

BY THE END YOU CAN

- Find indefinite integrals and use a boundary condition to find C.
- Evaluate definite integrals and find areas under a curve and between curves.
- Integrate e^x, 1/x, sin x, cos x and composites of the form f(ax + b), and use the reverse chain rule.
- Use calculus for displacement, velocity, acceleration and total distance.
- Integrate by substitution and by parts, including repeated parts. HL
- Use the standard arctan and arcsin integrals. HL
- Find volumes of revolution about the x-axis and y-axis. HL

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1 Indefinite integrals

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$$ FORMULA BOOKLET

add 1 to the power, divide by the new power

WORKED EXAMPLE

Find $\int(6x^2 - 4x + 3)\,dx$.

1. **$= 2x^3 - 2x^2 + 3x + C$** A1 A1

WORKED EXAMPLE

$f'(x) = 3x^2 - 2$ and $f(1) = 4$. Find $f(x)$.

1. $f(x) = x^3 - 2x + C$ A1
2. $1 - 2 + C = 4 \Rightarrow C = 5$ M1
3. **$f(x) = x^3 - 2x + 5$** A1

- **Rewrite first:** $\frac{1}{x^3} = x^{-3}$, $\sqrt{x} = x^{½}$, and split fractions: $\frac{x^2+1}{x} = x + \frac{1}{x}$. You can't integrate a product or quotient term by term.

COMMON MISTAKE

Integrating $(x + 1)(x - 2)$ as "$(x^2/2 + x)(x^2/2 - 2x)$". Expand the brackets first: $x^2 - x - 2$.

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2 Standard integrals

f(x)	∫ f(x) dx
e^x	$e^x + C$
$\frac{1}{x}$	$\ln \lvert x \rvert + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$f(ax + b)$	$\frac{1}{a} F(ax + b) + C$

- All of these are in the formula booklet. The last row is the "divide by a" rule: $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$, $\int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$, $\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln \lvert 2x+1 \rvert + C$.
- **Reverse chain rule:** if you can see a function and (a multiple of) its derivative, guess the answer and differentiate to check.

WORKED EXAMPLE

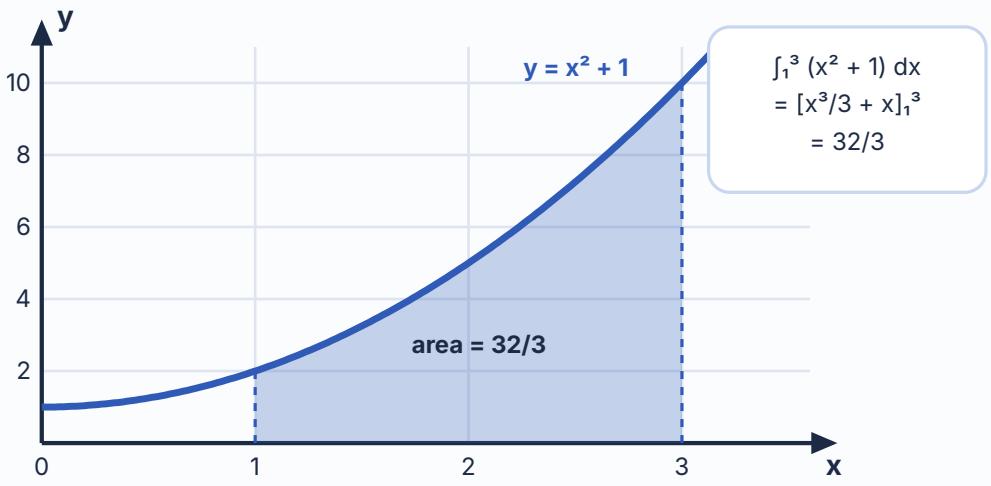
Find $\int x(x^2 + 1)^4\,dx$.

1. the derivative of $x^2 + 1$ is $2x$, so try $(x^2 + 1)^5$ M1
2. $d/dx\,(x^2 + 1)^5 = 10x(x^2 + 1)^4$, so divide by 10 M1
3. $\frac{(x^2 + 1)^5}{10} + C$ A1

EXAM TIP

$\int \frac{f'(x)}{f(x)}\,dx = \ln \lvert f(x) \rvert + C$ is the most useful reverse-chain pattern: $\int \frac{2x}{x^2 + 5}\,dx = \ln(x^2 + 5) + C$.

3 Definite integrals and area



The definite integral from 1 to 3 is the area under the curve between those x-values.

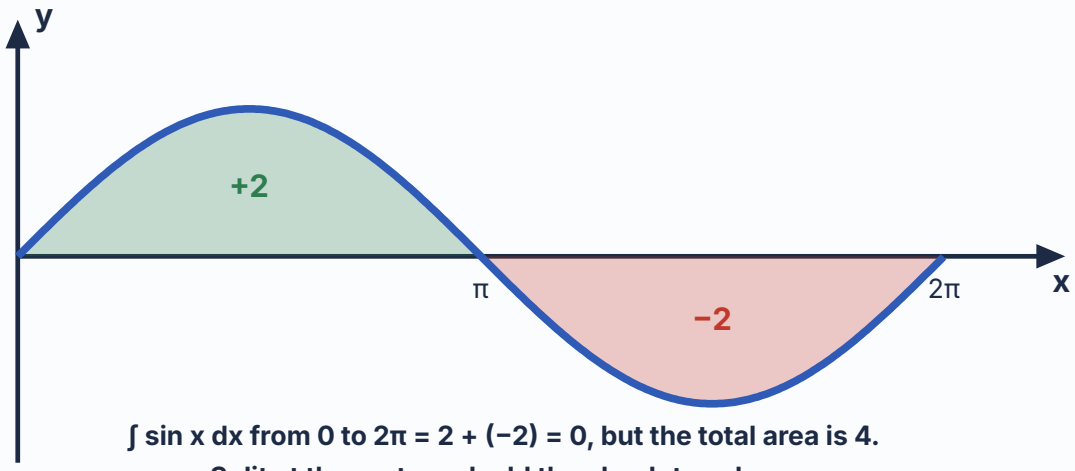
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

no + C needed

WORKED EXAMPLE

Find the area under $y = x^2 + 1$ between $x = 1$ and $x = 3$.

1. $[x^3/3 + x]_1^3$ **M1**
2. $= (9 + 3) - (\frac{1}{3} + 1)$ **M1**
3. $= \frac{32}{3}$ **A1**

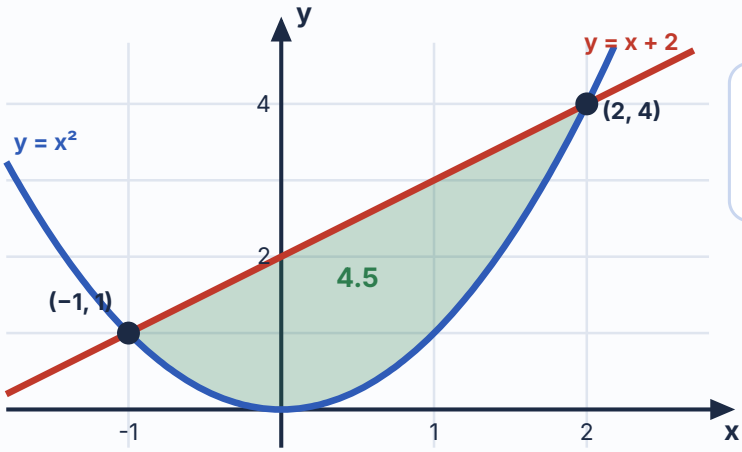


∫ sin x dx from 0 to 2π = 2 + (−2) = 0, but the total area is 4.
Split at the roots and add the absolute values.

An integral adds signed areas. To find a total area, split at the roots.

COMMON MISTAKE

Integrating straight across a root when asked for the area. Find where the curve crosses the axis, integrate each part separately, and add the absolute values.



∫ (top − bottom) dx between the intersections

Shade the region, decide which curve is on top, then integrate the difference.

WORKED EXAMPLE

Find the area enclosed by $y = x^2$ and $y = x + 2$.

1. $x^2 = x + 2 \Rightarrow x = -1$ or 2 M1 A1
2. $\int_{-1}^{2} (x + 2 - x^2)\,dx = [x^2/2 + 2x - x^3/3]_{-1}^{2}$ M1
3. $= \frac{10}{3} - (-\frac{7}{6}) = 4.5$ A1

GDC (CALCULATOR) METHOD

Paper 2: use the GDC's ∫ function (fnInt) for definite integrals. For an area, integrate |f(x)| directly; it does the splitting for you.

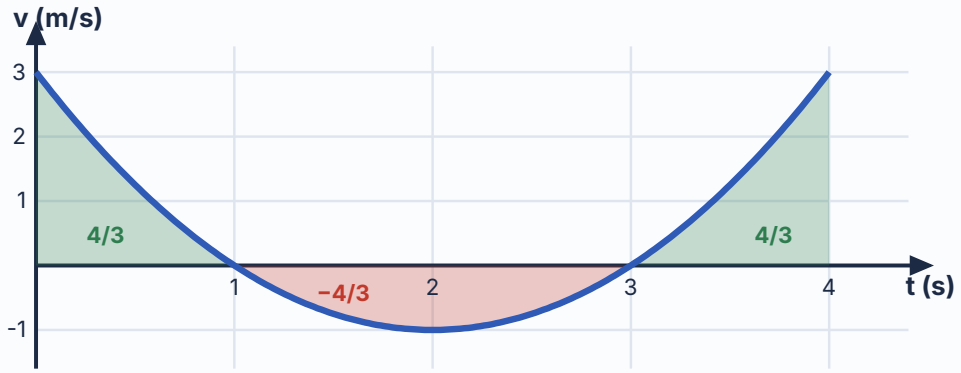
4 Kinematics

$$v = \frac{ds}{dt},\ a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

differentiate going down

$$\text{displacement} = \int_{t_1}^{t_2} v\,dt,\ \text{distance} = \int_{t_1}^{t_2} |v|\,dt$$ FORMULA BOOKLET

integrate going up



$v = t^2 - 4t + 3$: displacement = $\int v\,dt = 4/3$ m, distance = $\int |v|\,dt = 4$ m

Below the axis the particle is moving backwards.

Area above the axis is forward motion, below is backward. Displacement adds them with signs; distance adds their sizes.

WORKED EXAMPLE

A particle has velocity $v = t^2 - 4t + 3$ m s^{-1} for $0 \le t \le 4$. Find its displacement and the total distance travelled.

1. displacement = $\int_0^4 v\,dt = \frac{4}{3}$ m M1 A1
2. $v = 0$ at $t = 1$ and $t = 3$, so split there R1
3. **distance = $\frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$ m** A1

- **Speed** is |v|. The particle changes direction where v changes sign, not just where v = 0 (a double root of v doesn't count).
- **Speeding up or slowing down?** Speeding up when v and a have the same sign; slowing down when they have opposite signs.

5 Integration by substitution HL

- **Method:** choose u (usually the inside of a bracket or root), find du/dx, replace every x and dx, integrate in u, then substitute back. For a definite integral, change the limits to u-values instead.

WORKED EXAMPLE

Find $\int x\sqrt{(x+1)}\,dx$.

1. let $u = x + 1$, so $x = u - 1$ and $dx = du$ M1
2. $\int (u-1)u^{1/2}\,du = \int (u^{3/2} - u^{1/2})\,du$ M1
3. $= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$ A1 A1

6 Integration by parts HL

$$\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$$ FORMULA BOOKLET

choose u to be the part that gets simpler when differentiated

WORKED EXAMPLE

Find $\int xe^x\,dx$.

1. $u = x$, $dv/dx = e^x \Rightarrow v = e^x$ M1
2. $= xe^x - \int e^x\,dx$ A1
3. **$= xe^x - e^x + C$** A1

WORKED EXAMPLE

Find $\int x \ln x\,dx$.

1. $u = \ln x$ (ln is easier to differentiate than to integrate), $dv/dx = x \Rightarrow v = \frac{x^2}{2}$ M1
2. $= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$ A1
3. **$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$** A1

MEMORY TRICK

Choosing u: **LIATE**, which stands for Logs, Inverse trig, Algebraic, Trig, Exponential. Whichever comes first is u.

- **Repeated parts:** $\int x^2e^x\,dx$ needs parts twice. For $\int e^x \sin x\,dx$, use parts twice until the original integral reappears, then solve for it.

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7 arctan and arcsin integrals HL

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$ FORMULA BOOKLET

complete the square first if needed

$$\int \frac{1}{\sqrt{(a^2 - x^2)}}\,dx = \arcsin\left(\frac{x}{a}\right) + C,\ |x| < a$$ FORMULA BOOKLET

note: no 1/a in front this time

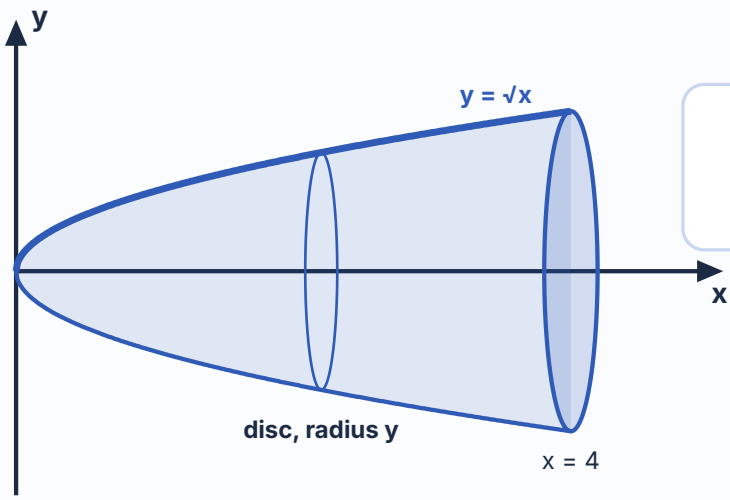
WORKED EXAMPLE

Find $\int_0^3 \frac{1}{x^2 + 9}\,dx$.

1. $= [\tfrac{1}{3}\arctan(x/3)]_0^3$ M1
2. $= \frac{1}{3} \times \frac{\pi}{4} = \frac{\pi}{12}$ A1

- **Completing the square** turns $\int \frac{1}{x^2 + 4x + 13}\,dx$ into $\int \frac{1}{(x + 2)^2 + 9}\,dx = \frac{1}{3}\arctan\left(\frac{x + 2}{3}\right) + C$.

8 Volumes of revolution HL



$V = \pi\int y^2\,dx$
$= \pi\int_0^4 x\,dx$
$= 8\pi$

Rotate the region 360° about the x-axis: each slice is a disc of area πy^2.

Each thin slice is a disc of radius y and thickness dx.

$V = \int_a^b \pi y^2\,dx$ (about the x-axis), $V = \int_a^b \pi x^2\,dy$ (about the y-axis) FORMULA BOOKLET

square the radius before integrating

WORKED EXAMPLE

The region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ is rotated 360° about the x-axis. Find the volume.

1. $V = \pi\int_0^4 x\,dx$ M1
2. $= \pi[x^2/2]_0^4 = 8\pi$ A1

WORKED EXAMPLE

The region between $y = x^2$, the y-axis and the line $y = 4$ ($x \geq 0$) is rotated about the y-axis. Find the volume.

1. $x^2 = y$, so $V = \pi\int_0^4 y\,dy$ M1
2. $= 8\pi$ A1

COMMON MISTAKE

Writing $\pi\int y\,dx$ instead of $\pi\int y^2\,dx$. The disc area is πr^2: square y first.

LINKING QUESTION

Integration is the reverse of Differentiation, and differential equations (Differential equations and Maclaurin series) are solved by integrating. Areas under a pdf give probabilities (Random variables, binomial and normal).

IDEA FOR YOUR IA / EXPLORATION

Estimate the area of an irregular shape (a lake, a leaf) by fitting functions to its boundary and integrating, or find the volume of a real object (a vase, a bottle) as a volume of revolution and compare with a measured volume of water.

Test yourself — cover the answers first

QUICK CHECK

1. Find $\int x^{-2}\,dx$.
2. Find $\int \frac{4}{x}\,dx$.
3. Find $\int_0^1 e^x\,dx$.
4. Find $\int \sin 3x\,dx$.
5. What is the difference between distance and displacement?
6. HL In $\int x\cos x\,dx$, which part should be u?

EXAM-STYLE

7. A particle starts at the origin with velocity 2 m s^{-1}, and its acceleration is $a = 6t$ m s^{-2}. Find s when $t = 2$.
8. HL Find $\int \ln x\,dx$.

Show answers

1. $-1/x + C$
2. $4 \ln |x| + C$
3. $e - 1$
4. $-\frac{1}{3}\cos 3x + C$
5. distance uses $|v|$; displacement can cancel
6. $u = x$
7. $v = 3t^2 + 2$
   $s = t^3 + 2t$ ($s = 0$ at $t = 0$)
   $s(2) = 12$ m
8. write it as $\int 1 \times \ln x \, dx$ with $u = \ln x$, $v = x$
   $= x \ln x - \int x \times \frac{1}{x} \, dx$
   $= x \ln x - x + C$

Stuck on a question?

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