IB DP MATHS AA · TOPIC 2 FUNCTIONS · SL 2.8 · AHL 2.12, 2.13, 2.15 · ~6 MIN READ

Polynomials, rational functions and inequalities

Reciprocal and rational functions with their asymptotes, and for HL: the factor and remainder theorems, polynomial graphs, sum and product of roots, oblique asymptotes, and solving polynomial and rational inequalities.

Reciprocal function | Rational functions & asymptotes | Factor theorem (HL) | Remainder theorem (HL) | Sum & product of roots (HL) | Oblique asymptotes (HL) | Inequalities (HL)

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No tag = **SL and HL** HL ONLY HL students only FORMULA BOOKLET given in the exam

M1 A1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

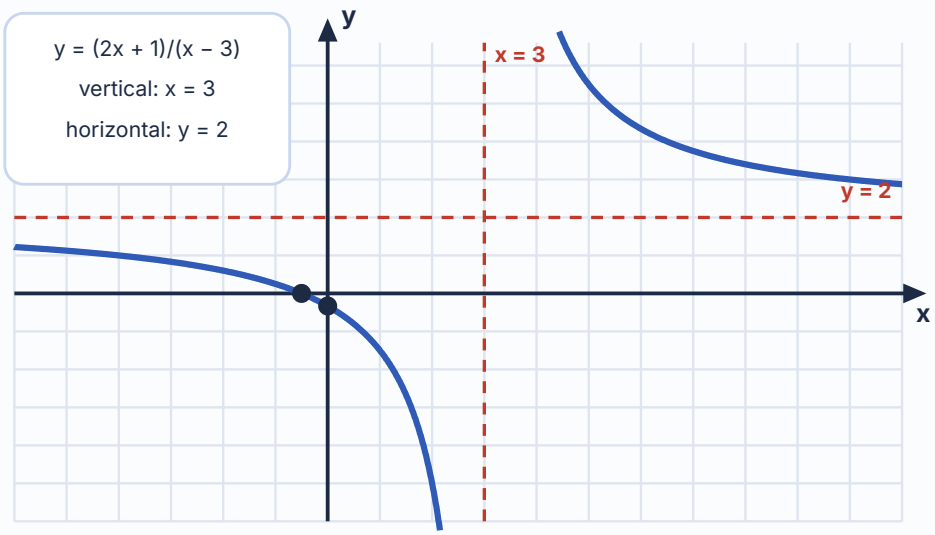
- A **rational function** $y = (ax + b)/(cx + d)$ has a **vertical asymptote** where the denominator is zero ($x = -d/c$) and a **horizontal asymptote** $y = a/c$.
- **HL factor theorem:** $(x - a)$ is a factor of $p(x) \Leftrightarrow p(a) = 0$. **Remainder theorem:** dividing $p(x)$ by $(x - a)$ leaves remainder $p(a)$.
- **HL:** for a polynomial $a_n x^n + \ldots + a_0$, the roots add to $-a_{n-1}/a_n$ and multiply to $(-1)^n a_0/a_n$.
- **HL inequalities:** find the critical values (roots, asymptotes), then test the sign between them; a sketch or sign chart beats algebra tricks.

BY THE END YOU CAN

- Sketch $y = 1/x$ and $y = (ax + b)/(cx + d)$, with intercepts and asymptotes.
- Use the factor and remainder theorems to factorise polynomials and find unknowns. HL
- Sketch polynomial graphs from their factors, including repeated roots. HL
- Use the sum and product of roots. HL
- Sketch rational functions with oblique asymptotes. HL
- Solve polynomial and rational inequalities, analytically and graphically. HL

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1 Reciprocal and rational functions



Two asymptotes split the curve into two branches.

- **y = 1/x** is self-inverse, with asymptotes x = 0 and y = 0. y = k/x is a stretch of it.
- **For y = $\frac{ax + b}{cx + d}$:** vertical asymptote x = −d/c; horizontal asymptote y = a/c; x-intercept where ax + b = 0; y-intercept b/d.

WORKED EXAMPLE

Sketch y = $\frac{2x + 1}{x - 3}$: give asymptotes and intercepts.

1. vertical **x = 3**, horizontal **y = 2** A1 A1
2. **x-intercept (−0.5, 0), y-intercept (0, −⅓)** A1

COMMON MISTAKE

Thinking the curve can never cross a horizontal asymptote. Horizontal asymptotes describe the behaviour as x → ±∞; for more complicated rational functions the curve can cross them in the middle.

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2 Polynomials HL

$$p(a) = 0 \Leftrightarrow (x - a) \text{ is a factor}$$

factor theorem

$$\text{remainder when } p(x) \div (x - a) = p(a)$$

remainder theorem; for (ax – b) the remainder is p(b/a)

WORKED EXAMPLE

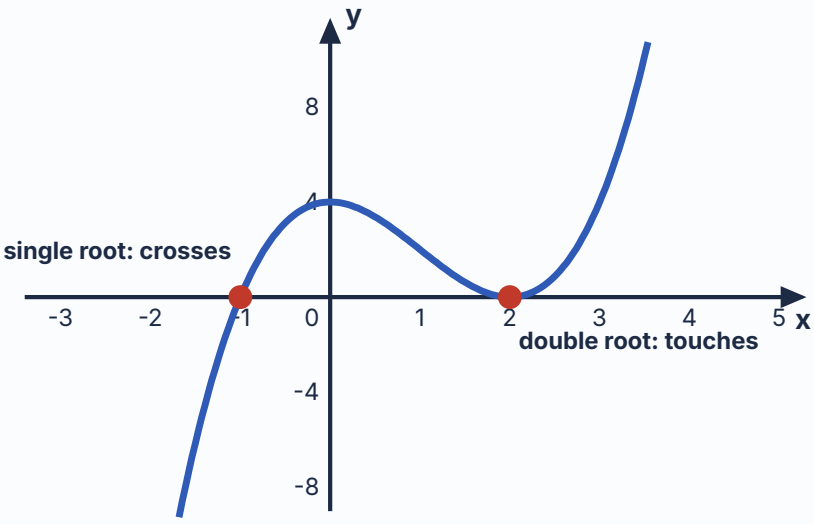
Factorise $p(x) = x^3 - 4x^2 + x + 6$ completely.

1. $p(-1) = -1 - 4 - 1 + 6 = 0$, so $(x + 1)$ is a factor M1
2. divide: $x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$ A1
3. **$= (x + 1)(x - 2)(x - 3)$** A1

WORKED EXAMPLE

When $2x^3 + ax^2 - 5$ is divided by $(x - 2)$ the remainder is 15. Find a.

1. $p(2) = 16 + 4a - 5 = 15$ M1
2. **$a = 1$** A1



$y = (x + 1)(x - 2)^2$: positive leading coefficient, so it rises to the right.

A single root crosses the axis; a double root touches it; the leading coefficient fixes the ends.

$$\text{sum of roots} = -\frac{a_{n-1}}{a_n}, \text{ product of roots} = \frac{(-1)^n a_0}{a_n}$$ FORMULA BOOKLET

for $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$

WORKED EXAMPLE

Find the sum and product of the roots of $2x^3 - 3x^2 + 4x - 5 = 0$.

1. sum $= -(-3)/2 = \frac{3}{2}$ A1
2. **product $= (-1)^3(-5)/2 = \frac{5}{2}$** A1

- **Complex roots count too:** a polynomial of degree n has exactly n roots (some may be complex, some repeated).

3 Rational functions with oblique asymptotes HL

- **Degree of top = degree of bottom + 1:** divide to find an oblique asymptote. $\frac{x^2 + 3x + 5}{x + 1} = x + 2 + \frac{3}{x + 1}$, so the asymptotes are $x = -1$ and **$y = x + 2$**.
- **Degree of top < degree of bottom** (e.g. $\frac{x}{x^2 - 4}$): horizontal asymptote $y = 0$, vertical asymptotes at each root of the denominator.

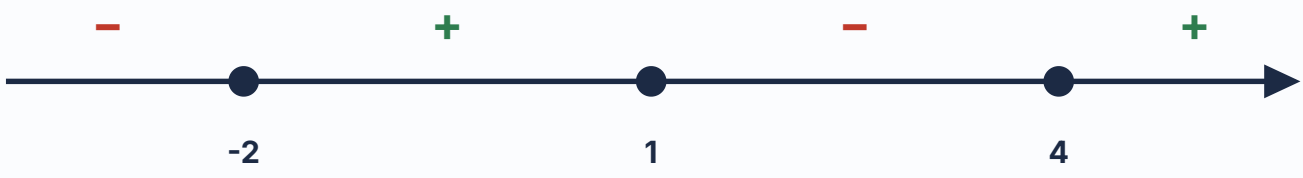
WORKED EXAMPLE

Find all asymptotes of $y = \frac{2x^2 - x + 1}{x - 1}$.

1. vertical **$x = 1$** A1
2. divide: $2x^2 - x + 1 = (x - 1)(2x + 1) + 2$ M1
3. **oblique $y = 2x + 1$** A1

4 Inequalities HL

Sign of $(x + 2)(x - 1)(x - 4)$



> 0 where the sign is +: $-2 < x < 1$ or $x > 4$

The sign can only change at a root (or an asymptote), so test one value in each interval.

WORKED EXAMPLE

Solve $(x + 2)(x - 1)(x - 4) > 0$.

1. critical values -2, 1, 4 A1
2. sign chart (or sketch of the cubic) M1
3. **$-2 < x < 1$ or $x > 4$** A1

WORKED EXAMPLE

Solve $\frac{x + 1}{x - 2} \leq 2$.

1. $\frac{x + 1}{x - 2} - 2 \leq 0 \Rightarrow \frac{5 - x}{x - 2} \leq 0$ M1
2. critical values $x = 2$ (excluded) and $x = 5$ A1
3. **$x < 2$ or $x \geq 5$** A1

COMMON MISTAKE

Multiplying both sides of an inequality by $(x - 2)$. Its sign is unknown, so the inequality might flip. Move everything to one side instead, or multiply by $(x - 2)^2$, which is positive.

GDC (CALCULATOR) METHOD

Paper 2: graph both sides and read the x-values where one is above the other. State the critical values to 3 s.f. and use the correct strict or non-strict signs.

LINKING QUESTION

Partial fractions (Proof, counting and partial fractions) undo the addition of rational functions, and the factor theorem connects to complex conjugate roots (Complex numbers).

IDEA FOR YOUR IA / EXPLORATION

Investigate how the shape of a rational function (asymptotes, turning points) changes with its parameters, or model cost per item, $y = (\text{fixed} + \text{variable} \times x)/x$, with a rational function.

Test yourself — cover the answers first

QUICK CHECK

1. State the asymptotes of $y = 3/(x + 1) - 2$.
2. Find the horizontal asymptote of $y = (4x - 1)/(2x + 5)$.
3. HL Is $(x - 2)$ a factor of $x^3 - 8$?
4. HL Find the remainder when $x^3 + 2x - 1$ is divided by $(x + 1)$.
5. HL What does a double root look like on a graph?
6. HL Solve $x^2(x - 3) \geq 0$.

EXAM-STYLE

7. HL $x = 2$ is a root of $x^3 + kx^2 - x - 2 = 0$. Find k and the other roots.
8. HL Solve $\frac{3}{x - 1} < \frac{2}{x}$.

Show answers

1. $x = -1$ and $y = -2$
2. $y = 2$
3. yes: $p(2) = 0$
4. $p(-1) = -4$
5. the curve touches the x-axis and turns back
6. $x = 0$ or $x \geq 3$
7. $8 + 4k - 2 - 2 = 0 \Rightarrow$ **$k = -1$**
   $x^3 - x^2 - x - 2 = (x - 2)(x^2 + x + 1)$
   other roots **$x = (-1 \pm \sqrt{3}\, i)/2$** (complex)
8. $3/(x - 1) - 2/x < 0 \Rightarrow \frac{x + 2}{x(x - 1)} < 0$
   critical values -2, 0, 1
   $x < -2$ or $0 < x < 1$

Stuck on a question?

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