IB DP MATHS AA · TOPIC 1 NUMBER AND ALGEBRA · SL 1.6 · AHL 1.10, 1.11, 1.15, 1.16 · ~8 MIN READ

Proof, counting and partial fractions

Deductive proof and identities, and for HL: proof by induction and by contradiction, counting principles with permutations and combinations, partial fractions, and solving systems of linear equations.

Deductive proof | LHS = RHS | Proof by induction (HL) | Proof by contradiction (HL) | Permutations & combinations (HL) | Partial fractions (HL) | Systems of equations (HL)

RS by Riyaz Sir
mentorriyaz.in

No tag = **SL and HL** HL ONLY HL students only FORMULA BOOKLET given in the exam

M1 A1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- A **proof** is a chain of steps where each one follows from the last. **Deductive proof** starts from known facts; an **identity** proof works one side until it equals the other.
- **HL proof by induction:** show the base case, assume case k, prove case k + 1, then write the conclusion.
- **HL proof by contradiction:** assume the opposite and reach something impossible.
- **HL counting:** multiply choices; **permutations** count orders, **combinations** ignore order.
- **HL partial fractions** split a fraction with factorised denominator into simpler fractions. **Systems** of up to three linear equations may have one, none or infinitely many solutions.
- **2029 syllabus:** proof by counterexample is removed at HL; everything else here stays.

BY THE END YOU CAN

- Write a clear deductive proof and prove identities (LHS ≡ RHS).
- Prove statements by mathematical induction. HL
- Use proof by contradiction. HL
- Count arrangements and selections with the multiplication principle, permutations and combinations. HL
- Split rational expressions into partial fractions. HL
- Solve systems of linear equations and interpret their solutions. HL

Riyaz Sir

1 Deductive proof

- **Good proof habits:** define your variables (let $n \in \mathbb{Z}$, so an odd number is $2n + 1$); show every step; end with a concluding sentence.

WORKED EXAMPLE

Prove that the sum of two consecutive odd numbers is a multiple of 4.

1. let the numbers be $2n + 1$ and $2n + 3$, $n \in \mathbb{Z}$ M1
2. sum $= 4n + 4 = 4(n + 1)$ A1
3. **$n + 1 \in \mathbb{Z}$, so the sum is a multiple of 4** R1

WORKED EXAMPLE

Prove that $(x + 1)^3 - (x - 1)^3 \equiv 6x^2 + 2$.

1. LHS $= (x^3 + 3x^2 + 3x + 1) - (x^3 - 3x^2 + 3x - 1)$ M1
2. $= 6x^2 + 2 =$ RHS A1
3. **so the identity holds for all x** R1

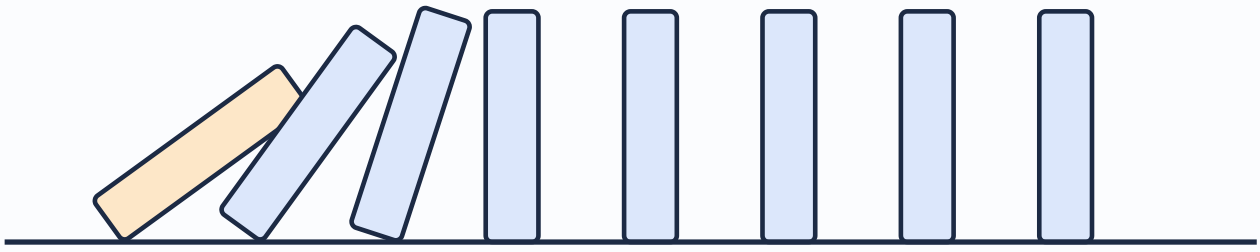
COMMON MISTAKE

Starting from the result and working "both sides" to 1 = 1. Start from ONE side (usually the more complicated one) and transform it into the other.

SYLLABUS NOTE

Proof by counterexample is removed from AA HL for the 2029 syllabus. It remains in the current syllabus (exams to November 2028): one example that breaks a statement disproves it.

2 Proof by induction HL



base case
n = 1 true

inductive step: if case k falls,
case k + 1 falls

Both together: every domino falls, so the statement is true for all $n \in \mathbb{Z}^+$.

Knock the first domino over, and show each one knocks the next: then all of them fall.

- **The four parts:** (1) **basis:** show it is true for n = 1; (2) **assumption:** assume it is true for n = k; (3) **inductive step:** use the assumption to show it is true for n = k + 1; (4) **conclusion:** "since true for n = 1, and true for k ⇒ true for k + 1, it is true for all n ∈ $\mathbb{Z}^+$".

WORKED EXAMPLE

Prove by induction that $\Sigma_{r=1}^{n} r = \frac{n(n+1)}{2}$.

1. n = 1: LHS = 1, RHS = 1 × 2/2 = 1 ✓ **A1**
2. assume $\Sigma_{r=1}^{k} r = k(k+1)/2$ **M1**
3. $\Sigma_{r=1}^{k+1} r = k(k+1)/2 + (k+1) = (k+1)(k+2)/2$ **M1** **A1**
4. **this is the formula with n = k + 1, so true for all n ∈ $\mathbb{Z}^+$ by induction** **R1**

WORKED EXAMPLE

Prove that $3^{2n} - 1$ is divisible by 8 for all n ∈ $\mathbb{Z}^+$.

1. n = 1: 9 − 1 = 8 ✓ **A1**
2. assume $3^{2k} - 1 = 8m$, m ∈ $\mathbb{Z}$ **M1**
3. $3^{2k+2} - 1 = 9 \times 3^{2k} - 1 = 9(8m+1) - 1 = 72m + 8 = 8(9m+1)$ **M1** **A1**
4. **divisible by 8, so true for all n by induction** **R1**

COMMON MISTAKE

Leaving out the conclusion sentence or the words "assume true for n = k". These are worth marks (R1) on their own.

3 Proof by contradiction HL

WORKED EXAMPLE

Prove that √2 is irrational.

1. assume √2 = p/q with p, q integers with no common factor **M1**
2. $2q^2 = p^2$, so p^2 is even, so p is even: p = 2m **A1**
3. $2q^2 = 4m^2 \Rightarrow q^2 = 2m^2$, so q is even too **A1**
4. **p and q share the factor 2: contradiction, so √2 is irrational** **R1**

Riyaz Sir

4 Counting principles HL

- **Multiplication principle:** if one choice can be made in m ways and a second in n ways, together in m × n ways.

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$ FORMULA BOOKLET

arrangements of r items chosen from n (order matters)

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$ FORMULA BOOKLET

selections of r items from n (order does not matter)

WORKED EXAMPLE

How many ways can 6 people sit in a row if two particular friends must sit together?

1. treat the pair as one unit: 5 units in 5! = 120 orders M1
2. the pair can swap: × 2 M1
3. **240** A1

WORKED EXAMPLE

A committee of 4 is chosen from 6 men and 5 women. How many committees have at least one woman?

1. total $^{11}C_4 = 330$ A1
2. no women: $^{6}C_4 = 15$ A1
3. **at least one woman: 330 − 15 = 315** A1

EXAM TIP

"At least one" is almost always quickest as **total – none**.

5 Partial fractions HL

WORKED EXAMPLE

Express $\frac{5x+1}{(x+1)(x-1)}$ in partial fractions.

1. $\frac{5x+1}{(x+1)(x-1)} \equiv \frac{A}{x+1} + \frac{B}{x-1}$ M1
2. $5x + 1 \equiv A(x-1) + B(x+1)$
3. $x = 1$: $6 = 2B \Rightarrow B = 3$; $x = -1$: $-4 = -2A \Rightarrow A = 2$ A1
4. $= \frac{2}{x+1} + \frac{3}{x-1}$ A1

- **Rules:** only for proper fractions (degree of top < degree of bottom) with distinct linear factors in the AA course. If the top's degree is too high, divide first.
- **Why bother:** partial fractions turn hard integrals into ln terms, and some series into sums that telescope.

6 Systems of linear equations HL

WORKED EXAMPLE

Solve $x + y + z = 6$, $2x - y + z = 3$, $x + 2y - z = 2$.

1. (1) + (3): $2x + 3y = 8$; (1) − (2): $-x + 2y = 3$ M1
2. so $x = 2y - 3$ and $2(2y - 3) + 3y = 8 \Rightarrow y = 2$ A1
3. **$x = 1$, $y = 2$, $z = 3$** A1

- **Three possibilities:** a unique solution (three planes meet at a point), no solution (inconsistent), or infinitely many (planes meet in a line): give the solution in terms of a parameter λ.

GDC (CALCULATOR) METHOD

Paper 2: use the simultaneous equation solver or row-reduce the matrix (rref). If the GDC reports "no solution" or "infinite solutions", say which and interpret it.

LINKING QUESTION

Induction proves the sigma formulae used in Sequences and the derivative patterns in Differentiation; partial fractions are the key step in several Integration questions.

IDEA FOR YOUR IA / EXPLORATION

Explore a counting problem with a real context (lottery odds, passwords, seating plans), or investigate a sequence conjectured from patterns and prove it by induction.

Test yourself — cover the answers first

QUICK CHECK

1. How many ways can 5 different books be arranged on a shelf?
2. Evaluate ${}^{10}P_3$ and ${}^{10}C_3$.
3. What are the four parts of an induction proof?
4. Write an odd integer and an even integer in general form.
5. HL How many ways can 3 boys and 3 girls stand in a line if boys and girls alternate?

EXAM-STYLE

6. HL Prove by induction that $n^3 - n$ is divisible by 6 for $n \in \mathbb{Z}^+$.
7. HL Express $\frac{x + 7}{x^2 + x - 2}$ in partial fractions.

Show answers

1. 120
2. 720 and 120
3. basis, assumption, inductive step, conclusion
4. $2n + 1$ and $2n$ ($n \in \mathbb{Z}$)
5. $2 \times 3! \times 3! = 72$
6. $n = 1$: $0 = 6 \times 0$ ✓
   $(k + 1)^3 - (k + 1) = (k^3 - k) + 3k(k + 1)$
   $k^3 - k = 6m$, and $k(k + 1)$ is even, so $3k(k + 1)$ is a multiple of 6
   so true for all n by induction
7. $x^2 + x - 2 = (x + 2)(x - 1)$
   $x + 7 \equiv A(x - 1) + B(x + 2)$
   $x = 1$: $B = 8/3$; $x = -2$: $A = -5/3$
   $\frac{8}{3(x - 1)} - \frac{5}{3(x + 2)}$

Stuck on a question?

Send it to me on WhatsApp. I read every message.

Want regular help? Send this to your parent →

Share with a friend

Ask Riyaz Sir