IB DP MATHS AA · TOPIC 4 STATISTICS AND PROBABILITY · SL 4.7–4.9, 4.12 · AHL 4.14 · ~8 MIN READ

Random variables, binomial and normal distributions

Discrete random variables and expected value, fair games, the binomial distribution with its mean and variance, the normal distribution, standardising with z-scores and inverse normal problems, and for HL: variance of discrete variables, E(aX + b), Var(aX + b) and continuous random variables with probability density functions.

Discrete random variables | Expected value | Fair games | Binomial distribution | Normal distribution | z-scores | Inverse normal | Variance & linear transformations (HL)

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No tag = **SL and HL** HL ONLY HL students only FORMULA BOOKLET given in the exam

M1 A1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

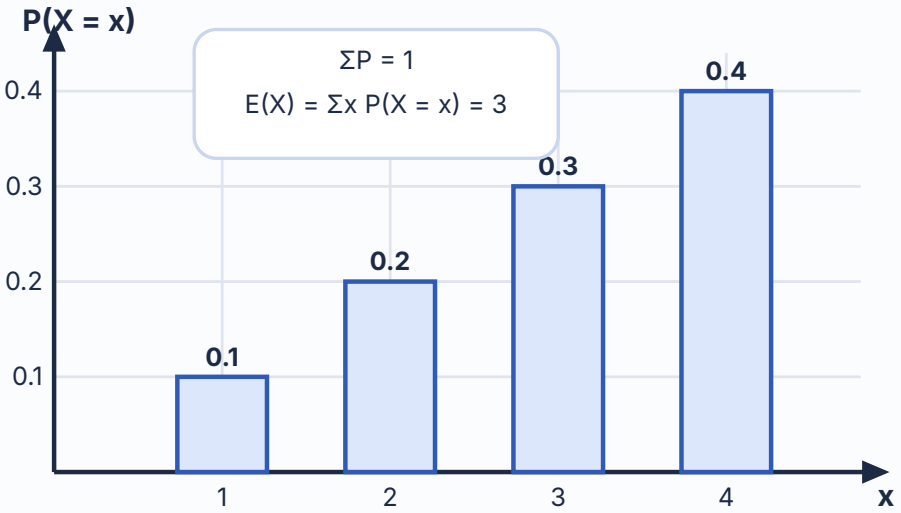
- A **discrete random variable** has a table of values and probabilities that **add to 1**. Its expected value is **E(X) = Σ x P(X = x)**.
- **Binomial X ~ B(n, p):** a fixed number n of independent trials, each a success with the same probability p. Mean np, variance np(1 − p).
- **Normal X ~ N(μ, σ²):** a symmetric bell curve for continuous data. Use the GDC for probabilities; use **z = (x − μ)/σ** when μ or σ is unknown.
- **HL:** $\text{Var}(X) = E(X^2) - [E(X)]^2$; $E(aX + b) = aE(X) + b$ and $\text{Var}(aX + b) = a^2\text{Var}(X)$; a continuous variable has a **pdf** f(x) with total area 1.

BY THE END YOU CAN

- Find probabilities and expected values for discrete random variables, and decide whether a game is fair.
- Recognise binomial situations and find binomial probabilities, mean and variance.
- Find normal probabilities and use inverse normal calculations.
- Standardise with z-scores to find an unknown mean or standard deviation.
- Find the variance of a discrete variable and use E(aX + b) and Var(aX + b). HL
- Use a pdf to find probabilities, the mean, variance, median and mode of a continuous random variable. HL

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1 Discrete random variables



The bars must add to 1; the expected value is the balance point.

$$E(X) = \Sigma\, x\, P(X = x)$$ FORMULA BOOKLET

the long-run average value

WORKED EXAMPLE

X takes the values 1, 2, 3, 4 with P(X = x) = kx. Find k and E(X).

1. $k + 2k + 3k + 4k = 1 \Rightarrow$ **k = 0.1** M1 A1
2. $E(X) = 1(0.1) + 2(0.2) + 3(0.3) + 4(0.4)$ M1
3. **= 3** A1

- **Fair game:** the expected gain (winnings minus the cost to play) is 0.

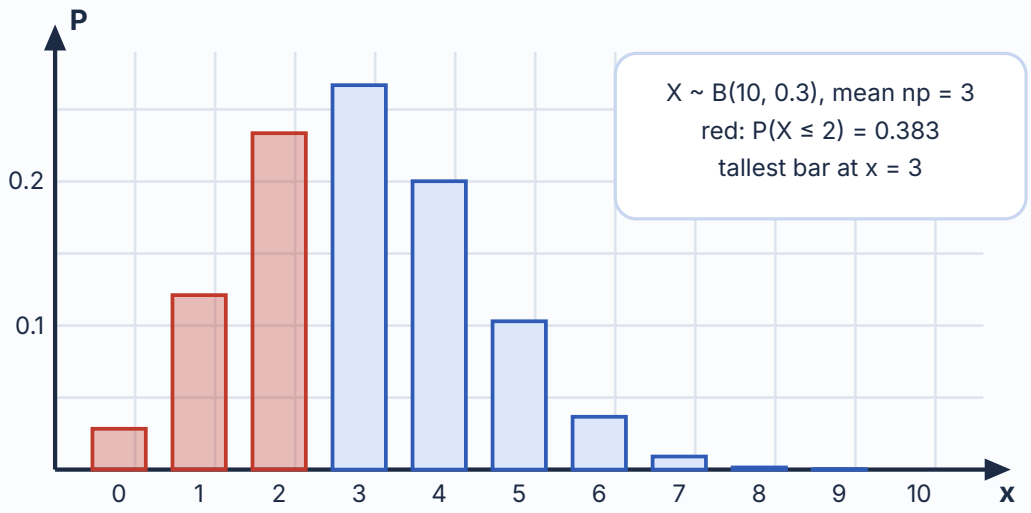
WORKED EXAMPLE

A game costs $3. You roll a fair die: a 6 wins $10, a 5 wins $5, anything else wins nothing. Is the game fair?

1. $E(\text{winnings}) = 10(\frac{1}{6}) + 5(\frac{1}{6}) = 2.5$ M1 A1
2. **E(gain) = 2.5 − 3 = −$0.50, so it is not fair (it favours the organiser)** R1

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2 The binomial distribution



B(10, 0.3): the shape peaks near the mean np = 3.

$$P(X = r) = {}^{n}C_{r}\, p^{r}(1-p)^{n-r}$$

r successes in n trials; in exams the GDC does this for you

$$E(X) = np,\ Var(X) = np(1-p)$$ FORMULA BOOKLET

mean and variance

- **The four conditions:** a fixed number of trials; two outcomes (success or failure); constant p; independent trials. Say which one fails if a situation isn't binomial.

WORKED EXAMPLE

X ~ B(10, 0.3). Find P(X = 3), P(X ≤ 2) and P(X ≥ 3).

1. $P(X = 3) = {}^{10}C_{3}(0.3)^{3}(0.7)^{7} =$ **0.267** A1
2. $P(X \le 2) =$ **0.383** (binomial cdf) A1
3. **$P(X \ge 3) = 1 - 0.383 = 0.617$** A1

WORKED EXAMPLE

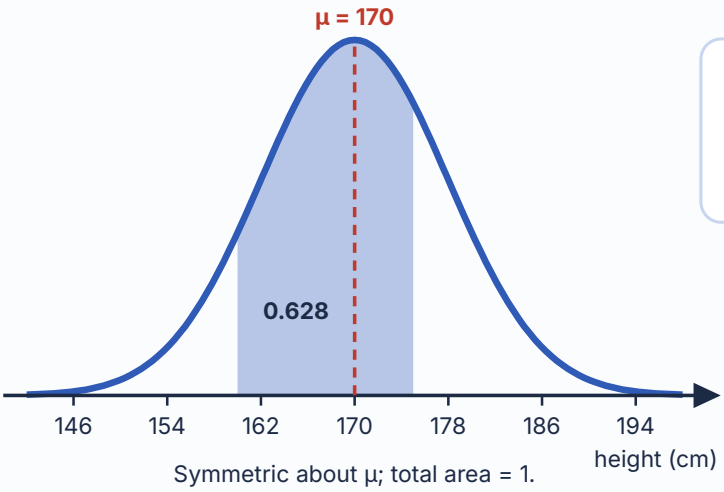
The probability that a seed grows is 0.2. Five seeds are planted. Find the probability that at least one grows.

1. $P(\text{none}) = 0.8^{5} = 0.32768$ M1
2. **$P(\text{at least one}) = 1 - 0.32768 = 0.672$** A1

COMMON MISTAKE

Treating "at least 3" as P(X > 3). For a discrete variable, P(X ≥ 3) = 1 − P(X ≤ 2). Write the inequality out every time before you press any buttons.

3 The normal distribution



$X \sim N(170, 8^2)$
$P(160 < X < 175)$
$= 0.628$
68% within $\mu \pm \sigma$

The shaded area is the probability. Sketch it every time: it stops most sign errors.

- **Facts:** symmetric about μ; about 68% of values are within $\mu \pm \sigma$, 95% within $\mu \pm 2\sigma$ and 99.7% within $\mu \pm 3\sigma$.

WORKED EXAMPLE

Heights are normally distributed, $X \sim N(170, 8^2)$ cm. Find $P(X > 180)$, $P(160 < X < 175)$ and the height exceeded by only 10% of people.

1. $P(X > 180) =$ **0.106** A1
2. $P(160 < X < 175) =$ **0.628** A1
3. **$P(X < k) = 0.9 \Rightarrow k = 180$ cm (inverse normal)** M1 A1

$$z = \frac{x - \mu}{\sigma}$$ FORMULA BOOKLET

standardising: $Z \sim N(0, 1)$

WORKED EXAMPLE

$X \sim N(60, \sigma^2)$ and $P(X < 50) = 0.2$. Find σ.

1. inverse normal: $z = -0.8416$ M1
2. $\frac{50 - 60}{\sigma} = -0.8416$ M1
3. **$\sigma = 11.9$** A1

EXAM TIP

The GDC's inverse normal works with the area to the **left** by default. For "the top 5%", use area 0.95 (or change the tail setting).

COMMON MISTAKE

Entering the variance instead of the standard deviation. N(170, 64) means $\sigma = 8$. The GDC asks for σ.

4 Variance and linear transformations HL

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$ FORMULA BOOKLET

$$E(X^2) = \Sigma\, x^2 P(X = x)$$

$$E(aX + b) = aE(X) + b,\ \text{Var}(aX + b) = a^2\text{Var}(X)$$ FORMULA BOOKLET

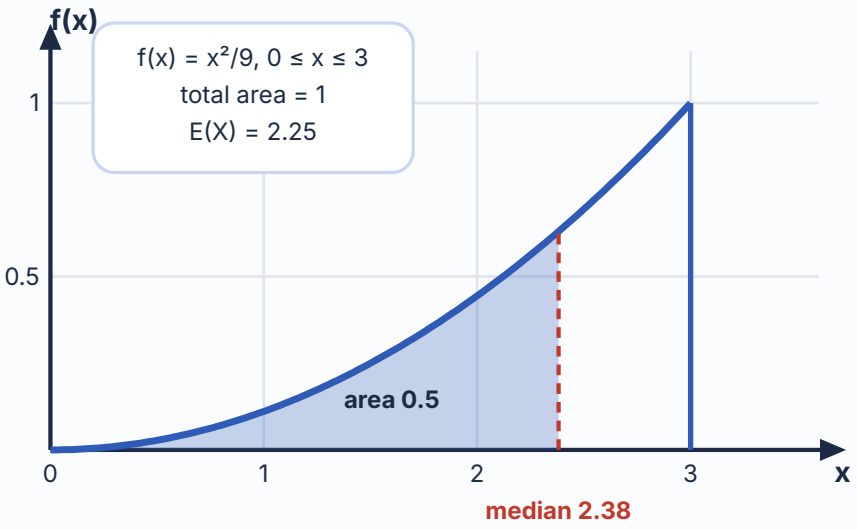
adding b moves the distribution but doesn't spread it

WORKED EXAMPLE

For X above (P(X = x) = 0.1x, x = 1, 2, 3, 4), find Var(X) and Var(5X − 2).

1. $E(X^2) = 0.1 + 0.8 + 2.7 + 6.4 = 10$ M1
2. $\text{Var}(X) = 10 - 3^2 =$ **1** A1
3. **Var(5X − 2) = 25 × 1 = 25** A1

5 Continuous random variables HL



Probability is area under the pdf. The median splits the area in half.

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$\int f(x)\,dx = 1$ over the domain, $P(a < X < b) = \int_a^b f(x)\,dx$

a pdf is never negative

$E(X) = \int x\,f(x)\,dx$, $Var(X) = \int x^2 f(x)\,dx - \mu^2$

FORMULA BOOKLET

same ideas as the discrete case, with $\int$ for Σ

WORKED EXAMPLE

$f(x) = kx^2$ for $0 \le x \le 3$ (and 0 elsewhere). Find k, E(X), the median and Var(X).

1. $\int_0^3 kx^2\,dx = 9k = 1 \Rightarrow$ **$k = \frac{1}{9}$** M1 A1
2. $E(X) = \int_0^3 \frac{x^3}{9}\,dx = \frac{81}{36} =$ **2.25** A1
3. median m: $\frac{m^3}{27} = 0.5 \Rightarrow$ **$m = 2.38$** M1 A1
4. **$E(X^2) = \int_0^3 \frac{x^4}{9}\,dx = 5.4$, so $Var(X) = 5.4 - 2.25^2 = 0.3375$** A1

- **Mode** of a continuous variable: where f(x) is greatest (the maximum of the pdf, which may be at an end of the domain). Here the mode is 3.

GDC (CALCULATOR) METHOD

Paper 2: use binompdf for P(X = r), binomcdf for P(X ≤ r), normalcdf for a range, and invNorm for a value. Write the distribution and the probability statement (for example X ~ B(10, 0.3), P(X ≤ 2)) before the answer.

LINKING QUESTION

The rules here rest on the probability ideas in Statistics and probability. Continuous random variables use definite integrals (Integration and kinematics), and the binomial coefficients are the nCr numbers from the binomial theorem (Sequences, series and the binomial theorem).

IDEA FOR YOUR IA / EXPLORATION

Test whether a real data set (heights, exam scores, reaction times) is close to normal by comparing its percentages within $\mu \pm \sigma$ and $\mu \pm 2\sigma$ with 68% and 95%, or model a real game of chance and compute its expected value.

Test yourself — cover the answers first

QUICK CHECK

1. E(X) of a fair six-sided die?
2. X ~ B(20, 0.4). Find E(X).
3. X ~ B(20, 0.4). Find Var(X).
4. X ~ N(50, 4^2). What percentage is roughly above 58?
5. Find z for x = 65 when $\mu = 50$ and $\sigma = 5$.
6. HL Var(X) = 4. Find Var(3X + 1).

EXAM-STYLE

7. Scores are normal with mean μ and SD σ. 10% score below 40 and 20% score above 70. Find μ and σ.
8. HL X has pdf $f(x) = \frac{3}{4}x(2 - x)$ for $0 \le x \le 2$. Find $P(X < 0.5)$ and state E(X) without integrating.

Show answers

1. 3.5
2. 8
3. 4.8
4. about 2.5%
5. 3
6. 36
7. $\frac{40-\mu}{\sigma} = -1.2816$ and $\frac{70-\mu}{\sigma} = 0.8416$
   subtract: $30 = 2.1232\sigma \Rightarrow$ **$\sigma = 14.1$**
   $\mu = 58.1$
8. $\int_0^{0.5} \frac{3}{4}(2x - x^2)\,dx = \frac{3}{4}(0.25 - \frac{1}{24}) =$ **$\frac{5}{32} = 0.156$**
   the pdf is symmetric about $x = 1$, so **$E(X) = 1$**

Stuck on a question?

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