IB DP MATHS AA · TOPIC 1 NUMBER AND ALGEBRA · SL 1.2, 1.3, 1.8, 1.9 · AHL 1.10 · ~8 MIN READ

Sequences, series and the binomial theorem

Arithmetic and geometric sequences and series, sigma notation, sums to infinity, the binomial theorem with Pascal's triangle and ${}^nC_r$, and (HL) the binomial series for negative and fractional powers.

Arithmetic sequences | Geometric sequences | Sigma notation | Sum to infinity | Binomial theorem | ${}^nC_r$ | Binomial series (HL)

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No tag = **SL and HL** HL ONLY HL students only FORMULA BOOKLET given in the exam

M1 A1 how marks are awarded highlight = comes up again and again Test yourself at the end

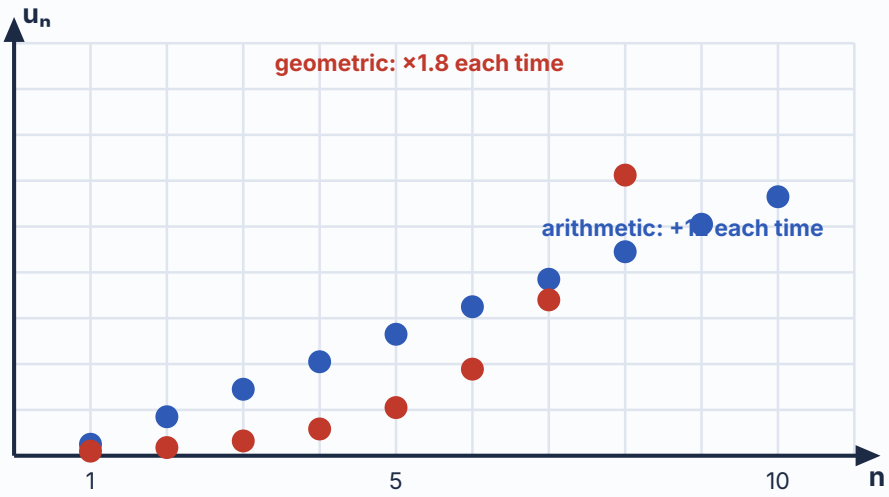
IN 30 SECONDS

- An **arithmetic** sequence adds the same **d** each time; a **geometric** sequence multiplies by the same **r**. Each has a formula for the nth term and for the sum of n terms, all in the formula booklet.
- A geometric series has a **sum to infinity** $u_1/(1 - r)$ only when $|r| < 1$.
- The **binomial theorem** expands $(a + b)^n$ using the coefficients ${}^nC_r$ (Pascal's triangle). The **general term** finds any single term without expanding everything.
- **HL:** for negative or fractional n, $(1 + x)^n$ becomes an infinite series, valid only for $|x| < 1$.
- **2029 syllabus:** financial applications (compound interest, annuities) are removed from AA. Everything else here is unchanged.

BY THE END YOU CAN

- Use the nth term and sum formulae for arithmetic and geometric sequences.
- Use sigma notation and solve problems for an unknown n.
- Find and use the sum to infinity of a convergent geometric series.
- Expand $(a + b)^n$ and find specific terms with ${}^nC_r$.
- Use the binomial series for any rational n and state its validity. HL

1 Arithmetic sequences and series



Arithmetic terms lie on a straight line; geometric terms grow (or shrink) by a constant ratio.

Plot the terms and the difference is obvious: a straight line (arithmetic) or a curve (geometric).

$$u_n = u_1 + (n-1)d$$ FORMULA BOOKLET

nth term

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$$ FORMULA BOOKLET

sum of the first n terms

WORKED EXAMPLE

An arithmetic sequence has $u_1 = 7$ and $d = 4$. Find u_{20} and S_{20}.

1. $u_{20} = 7 + 19 \times 4 = \mathbf{83}$ A1
2. $S_{20} = \frac{20}{2}(7 + 83) = \mathbf{900}$ M1 A1

WORKED EXAMPLE

For $3 + 7 + 11 + \ldots$, find the smallest n with $S_n > 500$.

1. $S_n = \frac{n}{2}(6 + 4(n-1)) = n(2n+1)$ M1
2. $2n^2 + n - 500 > 0 \Rightarrow n > 15.6$ (GDC or quadratic formula) M1
3. **$n = 16$** A1

COMMON MISTAKE

Giving n = 15.6. n counts terms, so it must be a positive integer: round **up** for "exceeds" questions.

2 Geometric sequences and series

$$u_n = u_1 r^{n-1}$$ FORMULA BOOKLET

nth term

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \; r \neq 1$$ FORMULA BOOKLET

sum of the first n terms

WORKED EXAMPLE

A geometric sequence starts 3, 6, 12, ... Find u_{10} and S_{10}.

1. $r = 2$ A1
2. $u_{10} = 3 \times 2^9 = \mathbf{1536}$ A1
3. $\mathbf{S_{10} = 3(2^{10} - 1)/(2 - 1) = 3069}$ A1

WORKED EXAMPLE

The 2nd term of a geometric sequence is 12 and the 5th is 96. Find u_1 and r.

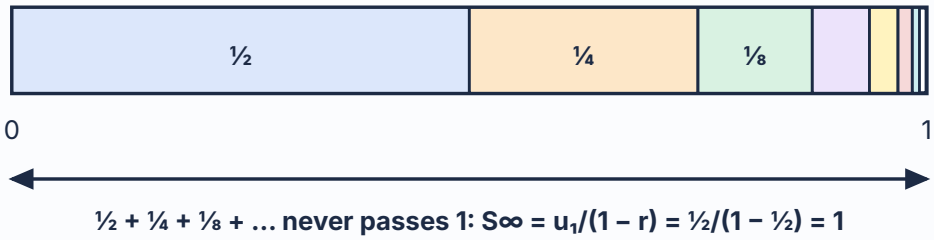
1. $u_1 r = 12$ and $u_1 r^4 = 96$ M1
2. divide: $r^3 = 8$, so **r = 2** A1
3. $\mathbf{u_1 = 6}$ A1

EXAM TIP

"Hence" after finding r usually means: substitute it straight into a sum formula. Divide one term equation by another to eliminate u_1.

Sum to infinity

An infinite geometric series converges when $|r| < 1$



$½ + ¼ + ⅛ + \ldots$ never passes 1: $S_\infty = u_1/(1 - r) = ½/(1 - ½) = 1$

If $|r| \geq 1$ the terms do not shrink, and the sum grows without limit (diverges).

$S_n = u_1(1 - r^n)/(1 - r)$: as $n \to \infty$, $r^n \to 0$ when $|r| < 1$.

Each bar fills half the gap that is left, so the total approaches 1 but never exceeds it.

$$S_\infty = \frac{u_1}{1-r},\ |r| < 1$$ FORMULA BOOKLET

sum to infinity of a convergent geometric series

WORKED EXAMPLE

Find the sum to infinity of 24 + 12 + 6 + ... and write 0.272727... as a fraction.

1. r = ½: $S_\infty = 24/(1 - ½) =$ **48** A1
2. 0.27 + 0.0027 + ...: $u_1 = 0.27$, r = 0.01 M1
3. $S_\infty = 0.27/0.99 = \frac{3}{11}$ A1

WORKED EXAMPLE

For which x does $1 + 2x + 4x^2 + ...$ converge, and what is its sum?

1. r = 2x, so we need $|2x| < 1$, i.e. $-½ < x < ½$ R1 A1
2. $S_\infty = \frac{1}{1-2x}$ A1

COMMON MISTAKE

Using S_∞ when $|r| \geq 1$. Always state "$|r| < 1$, so the series converges" before using the formula: it is often an R1 mark.

3 Sigma notation

- $\Sigma_{r=1}^{n} f(r)$ means: substitute r = 1, 2, ..., n and add. The number of terms is (top − bottom + 1).

WORKED EXAMPLE

Evaluate $\Sigma_{r=1}^{10} (3r + 2)$.

1. it is arithmetic: first term 5, last term 32, 10 terms M1
2. sum = $\frac{10}{2}(5 + 32) = 185$ A1

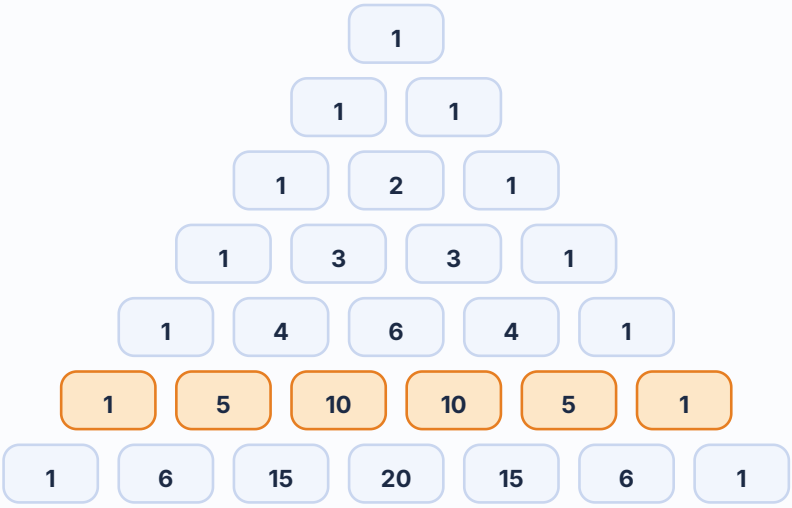
GDC (CALCULATOR) METHOD

Paper 2: use the sum(seq(...)) function (TI) or the Σ template (TI-Nspire, Casio) to evaluate sigma expressions directly, and a table of values to find the first n where a sum passes a target.

SYLLABUS NOTE

Financial applications (compound interest, depreciation, loans and annuities with the TVM solver) are removed from AA for the 2029 syllabus. Students sitting exams up to November 2028 still need them.

4 The binomial theorem



Each entry is the sum of the two above. Row n gives ${}^{n}C_{r}$, the coefficients of $(a + b)^n$.

Row 5 (orange) gives the coefficients of $(a + b)^5$: 1, 5, 10, 10, 5, 1.

$$(a + b)^n = a^n + {}^{n}C_1 a^{n-1}b + \ldots + {}^{n}C_r a^{n-r}b^r + \ldots + b^n$$ FORMULA BOOKLET

$n \in \mathbb{N}$

$${}^{n}C_r = \frac{n!}{r!(n-r)!}$$ FORMULA BOOKLET

the number of ways of choosing r items from n

- **General term:** $T_{r+1} = {}^{n}C_r a^{n-r}b^r$. Powers of a go down, powers of b go up, and they always add to n.

WORKED EXAMPLE

Find the coefficient of x^3 in $(2x - 3)^5$.

1. the term with x^3 is ${}^{5}C_2(2x)^3(-3)^2$ M1
2. $= 10 \times 8x^3 \times 9$ A1
3. **coefficient 720** A1

WORKED EXAMPLE

Find the constant term in $(x + \frac{2}{x^2})^6$.

1. general term ${}^{6}C_r x^{6-r}(2x^{-2})^r = {}^{6}C_r 2^r x^{6-3r}$ M1
2. constant: $6 - 3r = 0 \Rightarrow r = 2$ A1
3. **${}^{6}C_2 \times 2^2 = 60$** A1

COMMON MISTAKE

Forgetting to raise the coefficient: $(2x)^3 = 8x^3$, not $2x^3$. Put brackets round each part before you raise it.

5 The binomial series HL

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$ FORMULA BOOKLET

any rational n; infinite series, valid for $|x| < 1$

- **Not of the form (1 + x)?** Factor first: $(a + bx)^n = a^n(1 + \frac{b}{a}x)^n$, valid for $|bx/a| < 1$.

WORKED EXAMPLE

Expand $\sqrt{(4 + x)}$ up to x^2 and state where it is valid.

1. $\sqrt{(4 + x)} = 2(1 + \frac{x}{4})^{1/2}$ M1
2. $= 2(1 + \frac{1}{2}(\frac{x}{4}) - \frac{1}{8}(\frac{x}{4})^2 + \ldots)$ A1
3. **$= 2 + \frac{x}{4} - \frac{x^2}{64} + \ldots$, valid for $|x| < 4$** A1 A1

WORKED EXAMPLE

Use the expansion to estimate $\sqrt{4.2}$.

1. $x = 0.2$: $2 + 0.05 - 0.000625$ M1
2. **≈ 2.04938 (calculator: 2.04939)** A1

LINKING QUESTION

Geometric series reappear in the Maclaurin series note, where e^x and $\ln(1 + x)$ become infinite series, and in proof by induction, which can prove the sum formulae themselves.

IDEA FOR YOUR IA / EXPLORATION

Explore the convergence of different series (geometric, harmonic, $\Sigma 1/r^2$) and how fast their partial sums approach the limit; or model a real loan or a bouncing ball with a geometric series.

Test yourself — cover the answers first

QUICK CHECK

1. Find u_{12} for 5, 8, 11, ...
2. Find S_8 for 2, 6, 18, ...
3. Does $9 - 6 + 4 - \ldots$ converge? Find S_∞ if so.
4. Evaluate ${}^7C_3$.
5. How many terms are in $\Sigma_{r=4}^{20} r^2$?
6. HL For which x is the expansion of $(1 - 3x)^{-1}$ valid?

EXAM-STYLE

7. The sum of the first n terms of an arithmetic sequence is $S_n = 3n^2 + 2n$. Find u_1 and d.
8. A ball is dropped from 2.0 m and each bounce reaches 75 % of the previous height. Find the total distance travelled.
9. HL Find the first three terms of $(1 + 2x)^{-2}$ and use them to estimate $1/1.02^2$.

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Show answers

1. $5 + 11 \times 3 = 38$
2. $2(3^8 - 1)/2 = 6560$
3. $r = -\frac{2}{3}$, $|r| < 1$: $S_\infty = 9/(5/3) = 5.4$
4. 35
5. 17
6. $|x| < \frac{1}{3}$
7. $u_1 = S_1 = \mathbf{5}$
   $u_2 = S_2 - S_1 = 16 - 5 = 11$
   d = 6
8. down 2.0, then up and down $2 \times (1.5 + 1.125 + \ldots)$
   $= 2.0 + 2 \times 1.5/(1 - 0.75)$
   = **14 m**
9. $1 - 4x + 12x^2 - \ldots$
   $x = 0.01$: $1 - 0.04 + 0.0012$
   ≈ **0.9612**

Stuck on a question?

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Want regular help? Send this to your parent →

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