IB DP MATHS AA · TOPIC 3 GEOMETRY AND TRIGONOMETRY · SL 3.5–3.8 · AHL 3.9–3.11 · ~9 MIN READ

Trig identities, equations and graphs

The unit circle and exact values, the Pythagorean and double angle identities, graphs of sin, cos and tan with transformations, modelling with trig functions, solving trig equations in an interval, and for HL: sec, cosec, cot, inverse trig functions and compound angle identities.

Unit circle · Exact values · $\sin^2 + \cos^2 = 1$ · Double angle identities · Trig graphs · Amplitude & period · Trig models · Trig equations

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No tag = **SL and HL** · HL ONLY HL students only · FORMULA BOOKLET given in the exam
M1 A1 how marks are awarded · highlight = comes up again and again · Test yourself at the end

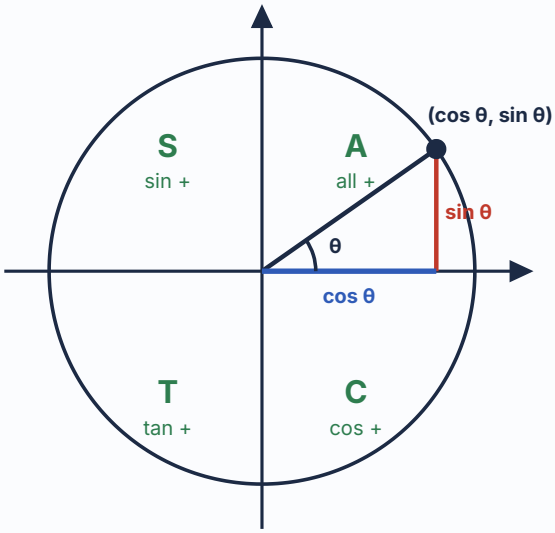
IN 30 SECONDS

- On the **unit circle**, the point at angle θ is (cos θ, sin θ). That one picture defines sin, cos and tan for every angle and gives their signs (CAST).
- **Key identities:** $\sin^2\theta + \cos^2\theta = 1$, $\tan\theta = \sin\theta/\cos\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$.
- **$y = a\sin(b(x - c)) + d$:** amplitude $|a|$, period $2\pi/b$, horizontal shift c, principal axis $y = d$.
- **Trig equations:** find one solution, then use symmetry and the period to find **every** solution in the given interval.
- **HL:** sec, cosec and cot, $1 + \tan^2\theta = \sec^2\theta$, compound angle formulas, and the inverse functions arcsin, arccos and arctan with their restricted ranges.

BY THE END YOU CAN

- Use the unit circle to find exact values and the signs of trig ratios in each quadrant.
- Use $\sin^2\theta + \cos^2\theta = 1$, $\tan\theta = \sin\theta/\cos\theta$ and the double angle identities.
- Sketch and transform the graphs of sin, cos and tan, and model periodic situations.
- Solve trig equations in a finite interval, including quadratic types.
- Use sec, cosec and cot, and the identities $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \operatorname{cosec}^2\theta$. HL
- Use the inverse trig functions and their domains and ranges. HL
- Use compound angle identities and prove trig identities. HL

1 The unit circle and exact values



radius 1
x = cos θ, y = sin θ
tan θ = sin θ/cos θ
CAST: which ratios are positive in each quadrant

The x-coordinate is cos θ, the y-coordinate is sin θ, and tan θ is the gradient of the radius.

θ	0	π/6 (30°)	π/4 (45°)	π/3 (60°)	π/2 (90°)
sin θ	0	½	√2/2	√3/2	1
cos θ	1	√3/2	√2/2	½	0
tan θ	0	√3/3	1	√3	undefined

MEMORY TRICK

Down the sin row the numbers are √0/2, √1/2, √2/2, √3/2, √4/2. The cos row is the same list backwards.

- **Symmetry rules:** sin(π − θ) = sin θ, cos(π − θ) = −cos θ, cos(−θ) = cos θ, sin(−θ) = −sin θ, and tan has period π.

WORKED EXAMPLE

Find the exact value of cos(4π/3).

1. 4π/3 is in the third quadrant, π/3 past π **M1**
2. cos is negative there **R1**
3. **cos(4π/3) = −½** **A1**

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2 Identities

$$\cos^2\theta + \sin^2\theta = 1,\ \tan\theta = \frac{\sin\theta}{\cos\theta}$$ FORMULA BOOKLET

Pythagorean identity

$$\sin 2\theta = 2\sin\theta\cos\theta$$ FORMULA BOOKLET

double angle

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$ FORMULA BOOKLET

pick the version that leaves one ratio

WORKED EXAMPLE

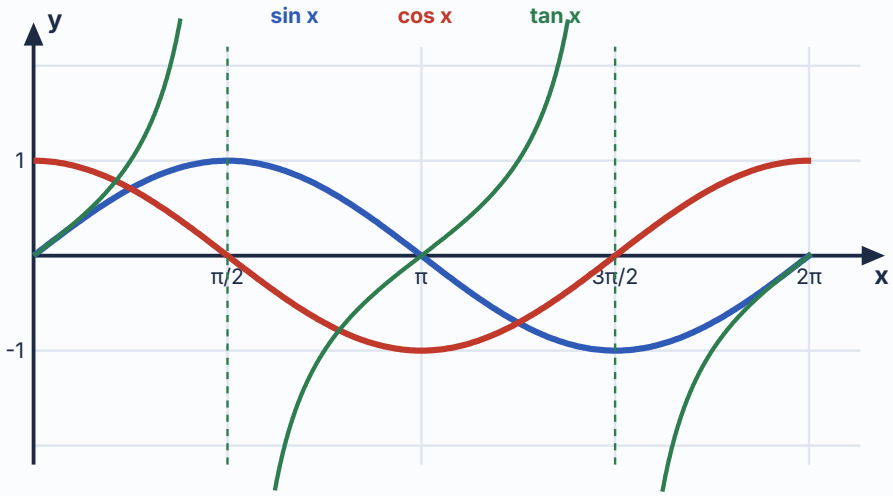
θ is obtuse and $\sin\theta = \frac{3}{5}$. Find $\cos\theta$ and $\sin 2\theta$.

1. $\cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$ M1
2. θ is obtuse, so **$\cos\theta = -\frac{4}{5}$** A1 R1
3. **$\sin 2\theta = 2(\frac{3}{5})(-\frac{4}{5}) = -\frac{24}{25}$** A1

COMMON MISTAKE

Forgetting the sign. A square root gives ±; the quadrant decides which. "Obtuse" means the second quadrant, where cos is negative.

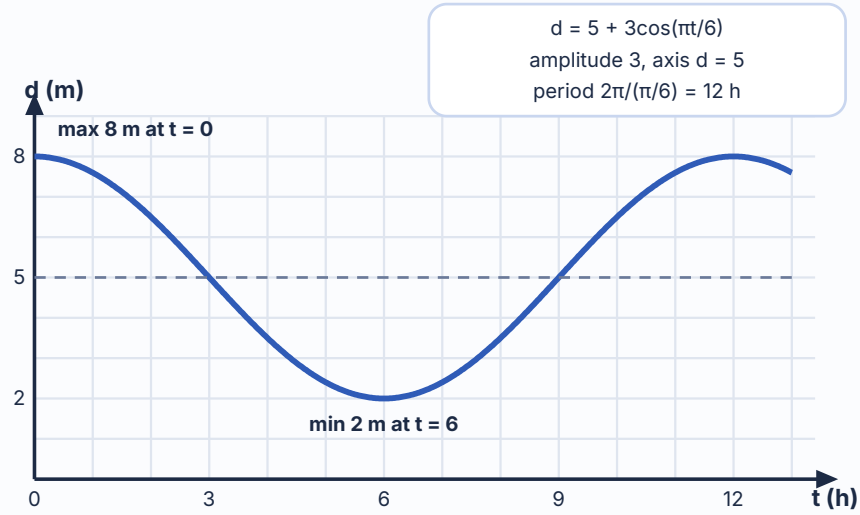
3 Graphs and models



sin and cos: period 2π, range [−1, 1]. tan: period π, asymptotes at x = π/2 + kπ.

sin and cos are the same wave shifted by π/2; tan repeats every π and has vertical asymptotes.

- **y = a sin(b(x − c)) + d:** amplitude $|a|$; period $\frac{2\pi}{b}$; shifted c to the right; principal axis y = d. The same applies to cos. For tan, the period is π/b.
- **Reading a model from data:** amplitude = (max − min)/2; axis d = (max + min)/2; period = time between two maxima, so b = 2π/period.



A tide model: maximum at t = 0, so a cosine curve fits without a shift.

WORKED EXAMPLE

The depth of water in a harbour is $d = 5 + 3\cos(\frac{\pi t}{6})$ metres, where t is hours after midnight, $0 \le t \le 12$. Find when the depth is 6.5 m.

1. $\cos(\frac{\pi t}{6}) = 0.5$ M1
2. $\frac{\pi t}{6} = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ A1
3. **t = 2 h or t = 10 h** A1

4 Solving trig equations

- **Method:** rearrange to sin x = k (or cos, tan); find the principal value; use the unit circle or the graph to find the second solution; then add or subtract periods until you leave the interval.
- **Change the interval for compound angles:** for sin 2x on $0 \le x \le 2\pi$, solve for 2x on $0 \le 2x \le 4\pi$, then halve.

WORKED EXAMPLE

Solve $2\cos^2 x - \cos x - 1 = 0$ for $0 \le x \le 2\pi$.

1. $(2\cos x + 1)(\cos x - 1) = 0$ M1
2. $\cos x = -\frac{1}{2}$ or $\cos x = 1$ A1
3. **$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$** A1 A1

WORKED EXAMPLE

Solve $\sin 2x = \cos x$ for $0 \le x \le 2\pi$.

1. $2 \sin x \cos x - \cos x = 0 \Rightarrow \cos x(2 \sin x - 1) = 0$ M1
2. $\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ A1
3. **$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$** A1

COMMON MISTAKE

Dividing both sides by cos x. You lose the solutions where cos x = 0. Factorise instead.

GDC (CALCULATOR) METHOD

Paper 2: graph both sides on the given interval and use intersect. Set the window to the interval exactly, and check the mode (radians or degrees) first.

5 Reciprocal trig functions HL

$$\sec\theta = \frac{1}{\cos\theta}, \operatorname{cosec}\theta = \frac{1}{\sin\theta}, \cot\theta = \frac{\cos\theta}{\sin\theta}$$

learn these; they are not given

$$1 + \tan^2\theta = \sec^2\theta, \; 1 + \cot^2\theta = \operatorname{cosec}^2\theta$$ FORMULA BOOKLET

divide $\sin^2 + \cos^2 = 1$ by $\cos^2\theta$ or $\sin^2\theta$

WORKED EXAMPLE

Solve $\sec^2 x = 3 \tan x - 1$ for $0 \le x \le \pi$.

1. $1 + \tan^2 x = 3 \tan x - 1 \Rightarrow \tan^2 x - 3 \tan x + 2 = 0$ M1
2. $\tan x = 1$ or 2 A1
3. **$x = \frac{\pi}{4}$ or $x = 1.11$** A1 A1

6 Inverse trig functions HL

Function	Domain	Range
arcsin x	$-1 \le x \le 1$	$-\pi/2 \le y \le \pi/2$
arccos x	$-1 \le x \le 1$	$0 \le y \le \pi$
arctan x	$x \in \mathbb{R}$	$-\pi/2 < y < \pi/2$

- sin x is not one-to-one, so its domain is restricted to $[-\pi/2, \pi/2]$ before it can have an inverse. The calculator always gives the value in the range above (the principal value).

WORKED EXAMPLE

Find the exact value of $\arcsin\left(-\frac{\sqrt{3}}{2}\right) + \arccos\left(-\frac{1}{2}\right)$.

1. $\arcsin(-\sqrt{3}/2) = -\frac{\pi}{3}$ A1
2. $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$ A1
3. **total $\frac{\pi}{3}$** A1

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7 Compound angle identities HL

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

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same signs

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

FORMULA BOOKLET

opposite signs

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

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watch the sign flip in the denominator

WORKED EXAMPLE

Find the exact value of tan 75°.

1. $\tan(45° + 30°) = \frac{1 + \sqrt{3}/3}{1 - \sqrt{3}/3}$ M1
2. $= \frac{3 + \sqrt{3}}{3 - \sqrt{3}} = \frac{(3 + \sqrt{3})^2}{6}$ M1
3. **$= 2 + \sqrt{3}$** A1

WORKED EXAMPLE

Prove that $\frac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta$.

1. LHS $= \frac{1 - (1 - 2\sin^2\theta)}{2 \sin\theta \cos\theta}$ M1
2. $= \frac{2\sin^2\theta}{2 \sin\theta \cos\theta} = \frac{\sin\theta}{\cos\theta}$ A1
3. **$= \tan\theta$ = RHS** AG

EXAM TIP

Proving an identity: start from the messier side and work towards the other. Never move terms across an "≡" as if it were an equation.

LINKING QUESTION

Exact values come straight from the 30-60-90 and 45-45-90 triangles (Trigonometry: triangles). The derivative of sin x is cos x only in radians (Differentiation), and complex numbers in polar form turn compound angle identities into simple multiplication (Complex numbers, HL).

IDEA FOR YOUR IA / EXPLORATION

Model daylight hours in your city over a year with y = a sin(b(t − c)) + d, using published sunrise and sunset times, and compare the fitted model with the data.

Test yourself — cover the answers first

QUICK CHECK

1. Find the exact value of sin(5π/6).
2. State the period of y = 3sin(2x).
3. State the amplitude and axis of y = 4 − 2cos x.
4. In which quadrant are sin negative and cos positive?
5. Write cos 2θ in terms of sin θ only.
6. HL Find sec(π/3).
7. HL State the range of arccos x.

EXAM-STYLE

8. Solve $2\sin(x + \frac{\pi}{4}) = \sqrt{3}$ for $0 \le x \le 2\pi$.
9. HL tan A = ½ and tan B = ⅓, with A and B acute. Show that A + B = $\frac{\pi}{4}$.

Show answers

1. ½
2. π
3. amplitude 2, axis $y = 4$
4. the fourth
5. $1 - 2\sin^2\theta$
6. 2
7. $0 \le y \le \pi$
8. $\sin(x + \pi/4) = \sqrt{3}/2$
   $x + \pi/4 = \pi/3$ or $2\pi/3$ (within $\pi/4 \le x + \pi/4 \le 9\pi/4$)
   $x = \frac{\pi}{12}$ or $\frac{5\pi}{12}$
9. $\tan(A + B) = \frac{½ + ⅓}{1 - ⅙} = \frac{5/6}{5/6} = 1$
   $A + B$ is between 0 and π, so $A + B = \frac{\pi}{4}$ AG

Stuck on a question?

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