IB DP MATHS AA · TOPIC 3 GEOMETRY AND TRIGONOMETRY · AHL 3.12–3.18 · ~10 MIN READ

Vectors, lines and planes

HL vectors in full: column and i, j, k notation, magnitude and unit vectors, the scalar product and angles, vector equations of lines, intersecting, parallel and skew lines, the vector product, equations of planes, and intersections and angles between lines and planes.

Vector notation | Magnitude & unit vectors | Position vectors | Scalar (dot) product | Vector equation of a line | Skew lines | Vector (cross) product | Equations of a plane

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No tag = **SL and HL** HL ONLY HL students only FORMULA BOOKLET given in the exam

M1 A1 how marks are awarded highlight = comes up again and again Test yourself at the end

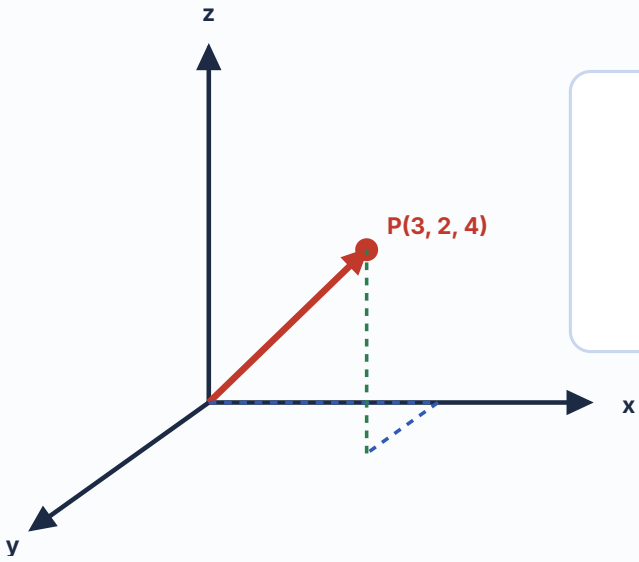
IN 30 SECONDS

- A **vector** has size and direction. **AB = b – a** (end minus start), and $|v| = \sqrt{(x^2 + y^2 + z^2)}$.
- **Scalar (dot) product:** $v \cdot w = v_1w_1 + v_2w_2 + v_3w_3 = |v||w| \cos \theta$. It is **0 exactly when the vectors are perpendicular**.
- **Line:** r = a + λb (point a, direction b). In 3D two lines are parallel, intersecting or **skew**.
- **Vector (cross) product:** v × w is perpendicular to both v and w, and $|v \times w| = |v||w| \sin \theta$ is the area of the parallelogram.
- **Plane:** r · n = a · n, or ax + by + cz = d, where n = (a, b, c) is the normal vector.
- This whole note is HL only (AHL 3.12–3.18) and unchanged for 2029.

BY THE END YOU CAN

- Use vector notation, find magnitudes and unit vectors, and work with position vectors. HL
- Use the scalar product to find angles and test for perpendicular vectors. HL
- Write vector, parametric and Cartesian equations of lines, and decide whether two lines are parallel, intersecting or skew. HL
- Find and use the vector product, including for areas. HL
- Write equations of planes in vector and Cartesian form. HL
- Find intersections of lines and planes and of two or three planes, and the angles between them. HL

1 Vector basics



OP = 3i + 2j + 4k

|OP| = √(9 + 4 + 16)

= √29

AB = b − a

(end − start)

A position vector goes from the origin O to a point; AB goes from A to B.

$$|v| = \sqrt{(v_1^2 + v_2^2 + v_3^2)}$$ FORMULA BOOKLET

magnitude

- **Unit vector** in the direction of v: v/|v|. **Parallel** vectors are multiples of each other: w = kv.
- **Notation:** (3, 2, 4) as a column vector is the same as 3i + 2j + 4k. Write vectors with an underline or arrow in your working.

WORKED EXAMPLE

A(1, 2, −1) and B(4, −2, 11). Find AB and a unit vector in the direction of AB.

1. AB = b − a = (3, −4, 12) A1
2. |AB| = √(9 + 16 + 144) = 13 A1
3. **unit vector $\frac{1}{13}(3, -4, 12)$** A1

- **Dividing a line:** the point P that divides AB in the ratio 1 : 2 has p = a + ⅓AB.

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2 The scalar product

$$\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3, \cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$$ FORMULA BOOKLET

angle between two vectors

WORKED EXAMPLE

Find the angle between a = (2, −1, 3) and b = (1, 4, −2).

1. $\mathbf{a} \cdot \mathbf{b} = 2 - 4 - 6 = -8$ A1
2. $\cos\theta = \frac{-8}{\sqrt{14}\sqrt{21}}$ M1
3. **θ = 118°** A1

WORKED EXAMPLE

Find p if (p, 2, −1) and (3, p, 4) are perpendicular.

1. $3p + 2p - 4 = 0$ M1
2. **p = 0.8** A1

EXAM TIP

Negative dot product means the angle is obtuse. For the angle between two **lines**, take the acute angle: use $|\mathbf{v} \cdot \mathbf{w}|$.

3 Equations of lines

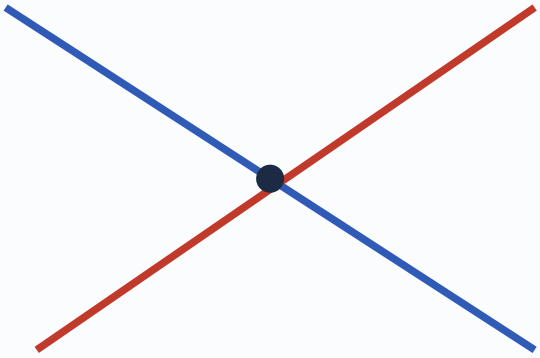
$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$ FORMULA BOOKLET

a = a point on the line, b = direction vector

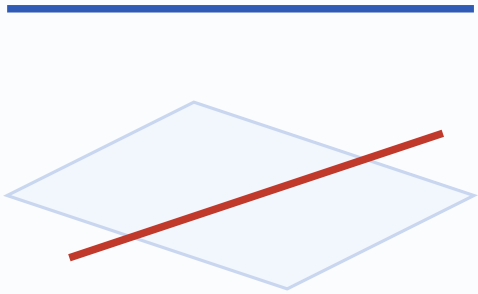
- **Parametric form:** $x = a_1 + \lambda b_1$, $y = a_2 + \lambda b_2$, $z = a_3 + \lambda b_3$. **Cartesian form:** $\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$.
- **Kinematics:** $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ describes a particle moving with constant velocity v; speed = $|\mathbf{v}|$. Two particles collide only if they are at the same point **at the same time**.

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Lines in 3D: parallel, intersecting, or skew



intersecting



skew: not parallel, never meet

In 2D, lines that are not parallel must meet. In 3D they can miss each other completely.

- **The test:** if the directions are multiples, the lines are parallel (or the same line). If not, equate the components: if one λ and μ satisfy all three equations they intersect; if not, they are **skew**.

WORKED EXAMPLE

L_1: r = (1, 2, 3) + λ(2, −1, 1) and L_2: r = (2, −3, 5) + μ(1, 1, 0). Show that they intersect and find the point.

1. x: $1 + 2\lambda = 2 + \mu$; y: $2 - \lambda = -3 + \mu$ M1
2. subtract: $3\lambda - 1 = 5 \Rightarrow \lambda = 2$, $\mu = 3$ A1
3. **z: 3 + 2 = 5 ✓, so they meet at (5, 0, 5)** R1 A1

WORKED EXAMPLE

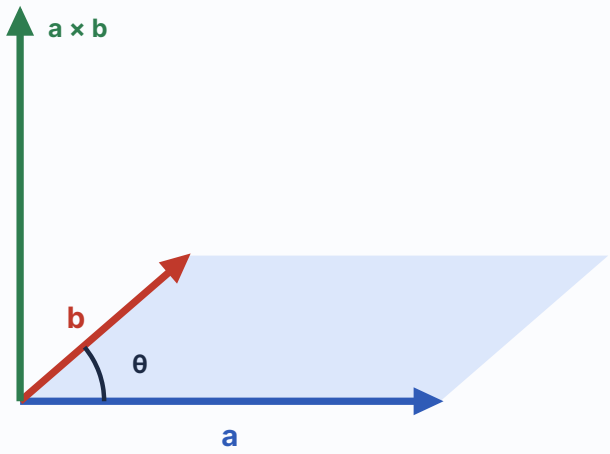
Find the acute angle between L_1 and L_2.

1. $\cos\theta = \frac{|2 - 1 + 0|}{\sqrt{6}\,\sqrt{2}} = \frac{1}{\sqrt{12}}$ M1
2. **θ = 73.2°** A1

COMMON MISTAKE

Checking only two of the three equations. Two equations always have a solution for λ and μ; the third one decides whether the lines meet.

4 The vector product



a × b ⊥ both a and b

$|a \times b| = |a||b| \sin\theta$

= area of the parallelogram

Use the right-hand rule: a along your first finger, b along your second, a × b along your thumb.

$$v \times w = (v_2w_3 - v_3w_2,\ v_3w_1 - v_1w_3,\ v_1w_2 - v_2w_1)$$ FORMULA BOOKLET

perpendicular to v and w

$$|v \times w| = |v||w| \sin\theta$$ FORMULA BOOKLET

area of parallelogram; half of it for a triangle

WORKED EXAMPLE

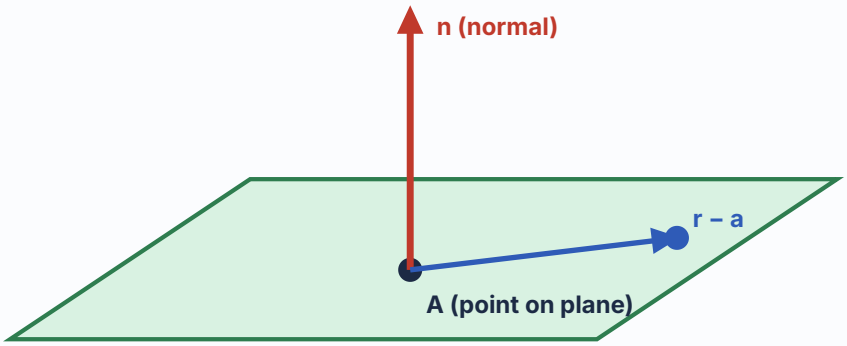
Find (2, −1, 3) × (1, 4, −2).

1. x: (−1)(−2) − (3)(4) = −10 M1
2. y: (3)(1) − (2)(−2) = 7; z: (2)(4) − (−1)(1) = 9 A1
3. **(−10, 7, 9)** A1

- **Check:** dot your answer with both original vectors. (−10, 7, 9) · (2, −1, 3) = −20 − 7 + 27 = 0 ✓.
- **Order matters:** w × v = −(v × w).

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5 Equations of planes



n ⊥ (r – a) for every point R in the plane, so r · n = a · n
Cartesian form: ax + by + cz = d, where n = (a, b, c)

Any vector lying in the plane is perpendicular to the normal n.

$r = a + \lambda b + \mu c$ FORMULA BOOKLET

vector form: a point and two directions in the plane

$r \cdot n = a \cdot n$, $ax + by + cz = d$ FORMULA BOOKLET

normal form and Cartesian form

WORKED EXAMPLE

Find the Cartesian equation of the plane through A(1, 0, 2), B(2, 1, 0) and C(0, 3, 1).

1. AB = (1, 1, −2), AC = (−1, 3, −1) A1
2. n = AB × AC = (5, 3, 4) M1 A1
3. **d = n · a = 5 + 0 + 8 = 13, so 5x + 3y + 4z = 13** A1

WORKED EXAMPLE

Find the area of triangle ABC.

1. area = ½|AB × AC| = ½√(25 + 9 + 16) M1
2. = $\frac{5\sqrt{2}}{2}$ ≈ 3.54 A1

6 Intersections and angles

- **Line meets plane:** substitute the parametric form of the line into the plane's equation and solve for λ.

WORKED EXAMPLE

Find where r = (1, 1, 1) + λ(1, 0, 2) meets the plane x + 2y − z = 6.

1. (1 + λ) + 2 − (1 + 2λ) = 6 M1
2. λ = −4 A1
3. **(−3, 1, −7)** A1

WORKED EXAMPLE

Find the acute angle between that line and the plane.

1. the angle with the normal: d · n = 1 + 0 − 2 = −1 M1
2. $\sin\theta = \frac{|-1|}{\sqrt{5}\,\sqrt{6}} = \frac{1}{\sqrt{30}}$ M1
3. **θ = 10.5°** A1

COMMON MISTAKE

Using cos instead of sin for a line and a plane. The dot product gives the angle with the **normal**, which is 90° minus the angle with the plane, so use sin θ.

- **Two planes:** their line of intersection has direction $n_1 \times n_2$. Find one point on it by setting one coordinate (for example z = 0) and solving the other two equations.

WORKED EXAMPLE

Find the line of intersection of x + y + z = 6 and x − y + 2z = 5.

1. direction = (1, 1, 1) × (1, −1, 2) = (3, −1, −2) M1 A1
2. z = 0: x + y = 6, x − y = 5 ⇒ (5.5, 0.5, 0) A1
3. **r = (5.5, 0.5, 0) + λ(3, −1, −2)** A1

- **Angle between two planes** = the angle between their normals (take the acute one). **Three planes** can meet at a point, along a line (a sheaf), or not at all; solve the 3 × 3 system with the GDC and interpret.

GDC (CALCULATOR) METHOD

Paper 2: the GDC solves 3 × 3 linear systems instantly. A "no solution" or "infinitely many solutions" message is your evidence of parallel planes or a line of intersection: say so in words.

LINKING QUESTION

Vectors turn the geometry of Trigonometry: triangles into algebra (the angle formula is the cosine rule in disguise). Velocity vectors link to Kinematics (Integration and kinematics), and complex numbers behave like 2D vectors under addition (Complex numbers).

IDEA FOR YOUR IA / EXPLORATION

Use vectors to model flight paths or drone routes in 3D and find the closest approach between two moving objects, or explore computer graphics: how 3D scenes are projected onto a 2D screen.

Test yourself — cover the answers first

QUICK CHECK

1. HL Find |(2, −3, 6)|.
2. HL Find (1, 2, 3) · (4, −1, 0).
3. HL What does a zero dot product tell you?
4. HL Give a normal vector to the plane 2x − y + 5z = 7.
5. HL What are skew lines?
6. HL Is (1, 0, 0) × (0, 1, 0) equal to k or −k?

EXAM-STYLE

7. HL Show that L_1: r = (1, 0, 0) + λ(1, 1, 0) and L_2: r = (0, 1, 3) + μ(0, 1, 1) are skew.
8. HL Find the acute angle between the planes 5x + 3y + 4z = 13 and x + y + z = 6.

Show answers

1. 7
2. 2
3. the vectors are perpendicular
4. (2, −1, 5)
5. lines that are not parallel and do not intersect
6. k
7. not parallel: (1, 1, 0) is not a multiple of (0, 1, 1)
   x: 1 + λ = 0 ⇒ λ = −1; y: −1 = 1 + μ ⇒ μ = −2
   z: 0 = 3 + (−2) = 1 ×, no common point, so **skew** AG
8. normals (5, 3, 4) and (1, 1, 1): dot product 12
   $\cos\theta = \frac{12}{\sqrt{50}\sqrt{3}}$
   θ = 11.5°

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