IB DP MATHS AI · TOPIC 5 CALCULUS · AHL 5.14–5.18 · ~8 MIN READ

Differential equations and phase portraits

HL differential equations for Maths AI: setting up models from rates of change, slope fields, separation of variables, Euler's method for single and coupled equations, coupled linear systems solved with eigenvalues and eigenvectors, phase portraits and their stability, and second-order equations written as systems.

Setting up DEs | Exponential growth & decay models | Slope fields | Separation of variables | Euler's method | Coupled systems | Eigenvalue solutions | Phase portraits

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No tag = **SL and HL** HL ONLY HL students only FORMULA BOOKLET given in the exam

M1 A1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- A **differential equation** describes how a quantity changes: $\frac{dP}{dt} = kP$ says the growth rate is proportional to the size.
- **Separation of variables:** write the DE as $g(y)\,dy = f(x)\,dx$, integrate both sides, then use the initial condition to find the constant.
- **Euler's method:** step along the slope field: $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h \times f(x_n, y_n)$.
- **Coupled linear systems** $\frac{dX}{dt} = MX$ have solutions $X = Ae^{\lambda_1 t}p_1 + Be^{\lambda_2 t}p_2$, where λ_1, λ_2 are the eigenvalues and p_1, p_2 the eigenvectors of M. The **eigenvalues** decide the shape of the **phase portrait**.
- This note is HL only and unchanged for 2029.

BY THE END YOU CAN

- Set up differential equations from a description of a rate of change. HL
- Sketch and interpret slope fields. HL
- Solve separable differential equations with an initial condition. HL
- Use Euler's method for first-order DEs and for coupled systems. HL
- Solve coupled linear systems with eigenvalues and eigenvectors, and sketch phase portraits. HL
- Classify equilibrium points and discuss stability. HL
- Write second-order DEs as coupled systems. HL

Riyaz Sir

1 Setting up and separating variables HL

- **From words:** "the population grows at a rate proportional to its size" ⇒ $\frac{dP}{dt} = kP$. "A drink cools at a rate proportional to the difference between its temperature and the room's" ⇒ $\frac{dT}{dt} = -k(T - T_{room})$.

WORKED EXAMPLE

A population satisfies $\frac{dP}{dt} = 0.03P$ with $P(0) = 1200$. Solve the DE, find $P(10)$, and find the doubling time.

1. $\int \frac{1}{P}\, dP = \int 0.03\, dt \Rightarrow \ln P = 0.03t + c$ M1 A1
2. **$P = 1200e^{0.03t}$** A1
3. **$P(10) = 1620$; doubling: $e^{0.03t} = 2 \Rightarrow t = 23.1$** A1 A1

WORKED EXAMPLE

Solve $\frac{dy}{dx} = \frac{x}{y}$ with $y(0) = 2$.

1. $\int y\, dy = \int x\, dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c$ M1 A1
2. $y(0) = 2 \Rightarrow y^2 = x^2 + 4$ M1
3. **$y = \sqrt{(x^2 + 4)}$ (positive root, since $y(0) > 0$)** A1

COMMON MISTAKE

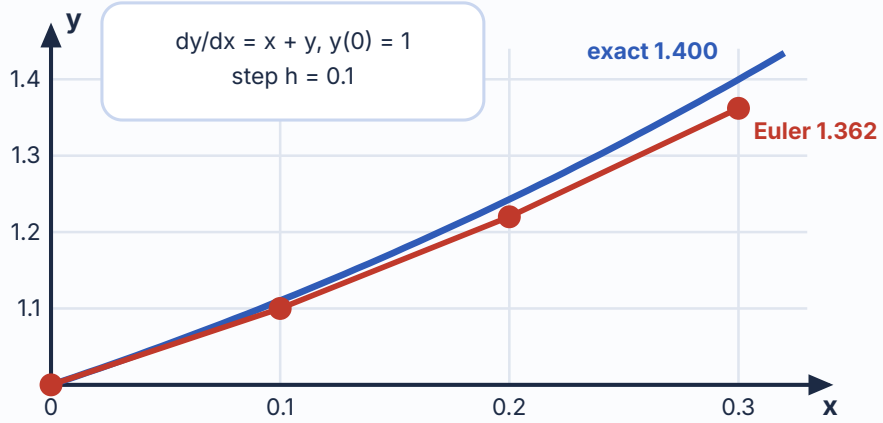
Adding the constant after exponentiating: $\ln P = 0.03t + c$ gives $P = e^c e^{0.03t} = Ae^{0.03t}$, not $e^{0.03t} + c$.

- **Slope fields:** at each point draw a short line with gradient $f(x, y)$. Solution curves follow the lines and never cross each other. Where $f = 0$ the lines are horizontal.

2 Euler's method HL

$$x_{n+1} = x_n + h,\ y_{n+1} = y_n + h \times f(x_n, y_n)$$ FORMULA BOOKLET

for $\frac{dy}{dx} = f(x, y)$



Euler's method follows the tangent for each step. Here the solution curve bends upwards, so Euler falls a little below it.

WORKED EXAMPLE

Use Euler's method with h = 0.1 to estimate y(0.3) for $\frac{dy}{dx} = x + y$, y(0) = 1.

1. $y_1 = 1 + 0.1(0 + 1) = 1.1$ M1
2. $y_2 = 1.1 + 0.1(0.1 + 1.1) = 1.22$ A1
3. **$y_3 = 1.22 + 0.1(0.2 + 1.22) = 1.362$** A1

- **Accuracy:** smaller steps give better estimates (the exact value here is 1.400). If the curve is concave up, Euler underestimates; concave down, it overestimates.

GDC (CALCULATOR) METHOD

Set up a table or spreadsheet with columns n, x_n, y_n and $y_n + h\,f(x_n, y_n)$, or use a recursive sequence. Show the first two steps by hand.

3 Coupled linear systems HL

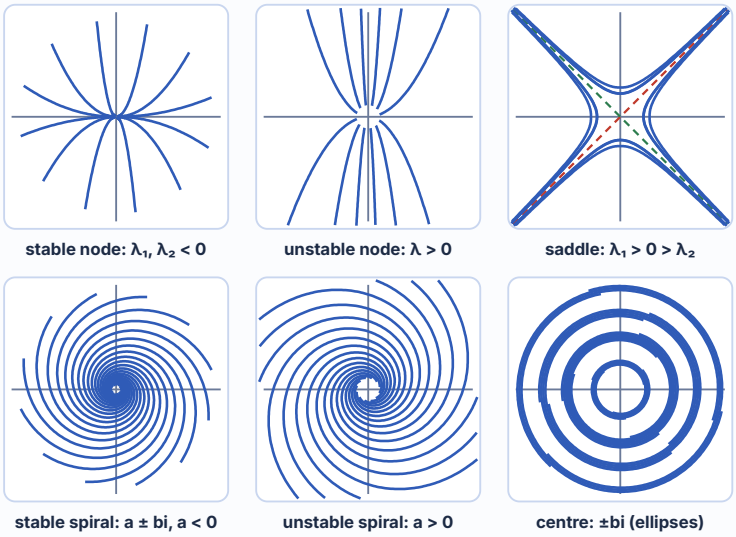
- **Form:** $\frac{dx}{dt} = ax + by$, $\frac{dy}{dt} = cx + dy$, or $\frac{dX}{dt} = MX$ with M = (a b; c d). They model interacting quantities: two competing species, two connected tanks.

WORKED EXAMPLE

Solve $\frac{dx}{dt} = x + 2y$, $\frac{dy}{dt} = 2x + y$ with x(0) = 3, y(0) = 1.

1. M = (1 2; 2 1): $\lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda = 3, -1$ M1 A1
2. eigenvectors (1; 1) for 3 and (1; −1) for −1 A1
3. $X = Ae^{3t}(1; 1) + Be^{-t}(1; -1)$; A + B = 3, A − B = 1 ⇒ A = 2, B = 1 M1
4. **$x = 2e^{3t} + e^{-t}$, $y = 2e^{3t} - e^{-t}$** A1

4 Phase portraits HL



The six phase portraits for $\frac{dX}{dt} = MX$. The eigenvalues of M decide which one you get.

Eigenvalues of M	Portrait	Equilibrium at origin
real, both negative	trajectories go into the origin	stable node
real, both positive	trajectories leave the origin	unstable node
real, opposite signs	in along one eigenvector, out along the other	saddle (unstable)
complex $a \pm bi$, $a < 0$	spiral inwards	stable spiral
complex $a \pm bi$, $a > 0$	spiral outwards	unstable spiral
purely imaginary $\pm bi$	closed loops (ellipses)	centre

- **Sketching:** draw the eigenvector lines first (for real eigenvalues), then add trajectories in the right directions. For spirals, check the direction at a point such as (1, 0): $\frac{dy}{dt} = c$ tells you whether it turns clockwise or anticlockwise.

WORKED EXAMPLE

Describe the phase portrait of the system above and the long-term behaviour from (3, 1).

1. $\lambda = 3$ and -1 have opposite signs: a **saddle** A1
2. **the e^{3t} term dominates, so the solution moves away along the eigenvector (1; 1): x and y both grow without limit, with $y/x \to 1$** R1 A1

COMMON MISTAKE

Mixing up the eigenvector lines. The **negative** eigenvalue's eigenvector is the line trajectories come **in** along; the **positive** one is the line they go **out** along.

5 Second-order differential equations HL

- **Method:** for $\frac{d^2x}{dt^2} = a\frac{dx}{dt} + bx$, let $y = \frac{dx}{dt}$. Then $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = bx + ay$: a coupled system with M = (0 1; b a).

WORKED EXAMPLE

Write $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$ as a system and describe the motion.

1. $y = \frac{dx}{dt}$: $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -2x - 3y$, M = (0 1; −2 −3) M1 A1
2. $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda = -1, -2$: both negative A1
3. **stable node: x returns to 0 without oscillating (an over-damped spring)** R1

- **Euler for systems:** step both variables together: $x_{n+1} = x_n + h\,f_1(x_n, y_n, t_n)$, $y_{n+1} = y_n + h\,f_2(x_n, y_n, t_n)$. This also works for non-linear systems such as predator–prey models.

LINKING QUESTION

Separation of variables needs Integration, areas and kinematics; coupled systems need eigenvalues and eigenvectors from Matrices and eigenvalues; exponential solutions are the models of Modelling with functions. For the AA treatment, with integrating factors, see Differential equations and Maclaurin series.

IDEA FOR YOUR IA / EXPLORATION

Model a real cooling curve or the spread of a rumour with a differential equation, fit the parameter from your data, and compare the model with measurements. Or simulate a predator–prey system with Euler's method in a spreadsheet and investigate its equilibrium.

Test yourself — cover the answers first

QUICK CHECK

1. HL Write "the rate of decay of mass m is proportional to m" as a DE.
2. HL What does a slope field line show at a point?
3. HL Give the Euler formula for y_{n+1}.
4. HL Eigenvalues −2 and −5. What is the equilibrium?
5. HL Eigenvalues 1 ± 2i. What does the portrait look like?
6. HL Write $x'' = -4x$ as a system.

EXAM-STYLE

7. HL A tank's water level h satisfies $\frac{dh}{dt} = -0.4\sqrt{h}$ with h(0) = 25. Solve the DE and find when the tank is empty.
8. HL For $\frac{dX}{dt}$ = (0 −1; 1 0)X, find the eigenvalues, describe the phase portrait, and give the direction of motion at (1, 0).

Show answers

1. $dm/dt = -km$
2. the gradient dy/dx there
3. $y_n + h\,f(x_n, y_n)$
4. a stable node
5. an unstable spiral
6. $x' = y$, $y' = -4x$
7. $\int h^{-½}\,dh = \int -0.4\,dt \Rightarrow 2\sqrt{h} = -0.4t + c$
   $c = 10$, so $\sqrt{h} = 5 - 0.2t$, **$h = (5 - 0.2t)^2$**
   empty when $\sqrt{h} = 0$: **$t = 25$**
8. $\lambda^2 + 1 = 0 \Rightarrow$ **$\lambda = \pm i$**
   a **centre**: trajectories are closed circles around the origin
   at (1, 0): $dx/dt = 0$, $dy/dt = 1$, so it moves upwards: **anticlockwise**

Stuck on a question?

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Ask Riyaz Sir