IB DP MATHS AI · TOPIC 5 CALCULUS · SL 5.1–5.4, 5.6, 5.7 · AHL 5.9, 5.10 · ~8 MIN READ

Differentiation and optimisation

The derivative as a rate of change and a gradient, differentiating ax^n, increasing and decreasing functions, tangents and normals, stationary points, and optimisation in context, plus for HL: derivatives of sin, cos, tan, e^x and ln x, the chain, product and quotient rules, related rates and the second derivative.

Derivative as rate of change · Limits & gradients · Differentiating ax^n · Increasing & decreasing · Tangents & normals · Stationary points · Local max & min · Optimisation

RS by Riyaz Sir
mentorriyaz.in

No tag = **SL and HL** · HL ONLY HL students only · FORMULA BOOKLET given in the exam

M1 A1 how marks are awarded · highlight = comes up again and again · Test yourself at the end

IN 30 SECONDS

- The **derivative** $f'(x) = \frac{dy}{dx}$ is the instantaneous rate of change of y with x, and the gradient of the tangent to the curve.
- **Power rule:** if $f(x) = ax^n$, then $f'(x) = anx^{n-1}$. Differentiate each term separately.
- **f′(x) > 0** means increasing; **f′(x) < 0** means decreasing; **f′(x) = 0** at stationary points (local maxima and minima).
- **Optimisation:** write the quantity as a function of one variable, set its derivative to 0 (or use the GDC's maximum/minimum), then check it is a maximum or minimum and answer in context.
- **HL:** derivatives of sin x, cos x, tan x, e^x and ln x; the chain, product and quotient rules; related rates; and the second derivative for concavity and classifying stationary points.

BY THE END YOU CAN

- Interpret the derivative as a rate of change and as a gradient, in context.
- Differentiate functions made of terms ax^n (n an integer).
- Find where functions are increasing or decreasing.
- Find equations of tangents and normals.
- Find and classify stationary points, and solve optimisation problems in context.
- Differentiate trig, exponential and logarithmic functions, and use the chain, product and quotient rules. HL
- Solve related rates problems. HL
- Use the second derivative to classify stationary points and describe concavity. HL

Riyaz Sir

1 The derivative

- **As a limit:** the gradient of the chord from x to x + h tends to the gradient of the tangent as h → 0. In AI you meet this as an idea, often from a GDC table of chord gradients.
- **In context:** if V(t) is the volume of water in a tank (litres) at time t (minutes), V′(t) is the rate of flow in litres per minute. A negative value means the tank is emptying.

$$f(x) = ax^n \Rightarrow f'(x) = anx^{n-1}$$ FORMULA BOOKLET

n is an integer at SL

WORKED EXAMPLE

Differentiate $f(x) = 4x^3 - 5x^2 + \frac{2}{x} - 7$.

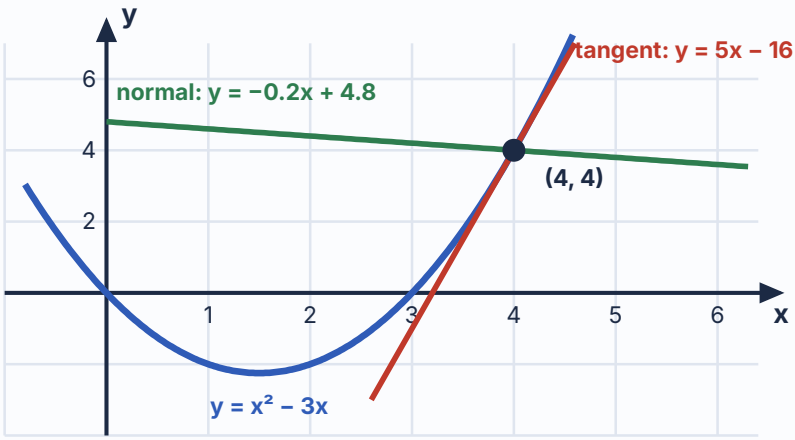
1. rewrite $\frac{2}{x}$ as $2x^{-1}$ M1

2. **$f'(x) = 12x^2 - 10x - 2x^{-2} = 12x^2 - 10x - \frac{2}{x^2}$** A1 A1

COMMON MISTAKE

Differentiating $\frac{2}{x}$ as "2". Rewrite fractions with negative powers first: $\frac{2}{x} = 2x^{-1} \Rightarrow -2x^{-2}$.

2 Tangents and normals



The tangent has the gradient of the curve at the point; the normal is perpendicular to it.

WORKED EXAMPLE

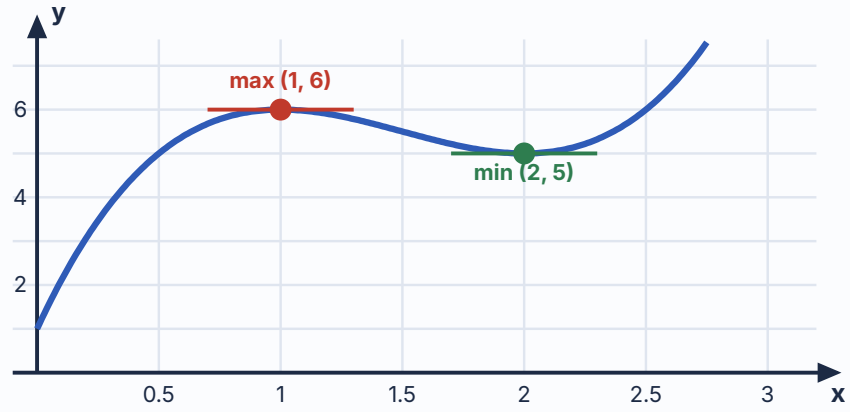
Find the equations of the tangent and normal to $y = x^2 - 3x$ at $x = 4$.

1. $y(4) = 4$; $\frac{dy}{dx} = 2x - 3 = 5$ at $x = 4$ A1 A1
2. tangent: $y - 4 = 5(x - 4)$, **$y = 5x - 16$** A1
3. **normal gradient $-\frac{1}{5}$: $y = -0.2x + 4.8$** A1

GDC (CALCULATOR) METHOD

The GDC's dy/dx (or nDeriv) tool gives the gradient at a point, and many GDCs draw the tangent line for you. Still write the gradient and the point you used.

3 Stationary points



sign of f′: **+ (rising)** **− (falling)** **+ (rising)**

$f(x) = 2x^3 - 9x^2 + 12x + 1$: $f'(x) = 0$ where the tangent is horizontal.

The sign of $f'(x)$ changes from + to − at a maximum and from − to + at a minimum.

WORKED EXAMPLE

$f(x) = 2x^3 - 9x^2 + 12x + 1$. Find and classify the stationary points, and state where f is decreasing.

1. $f'(x) = 6x^2 - 18x + 12 = 6(x - 1)(x - 2)$ A1
2. $f'(x) = 0$ at $x = 1$ and 2, giving **(1, 6)** and **(2, 5)** M1 A1
3. f' changes + to − at $x = 1$ (**maximum**) and − to + at $x = 2$ (**minimum**) R1
4. **decreasing for $1 < x < 2$** A1

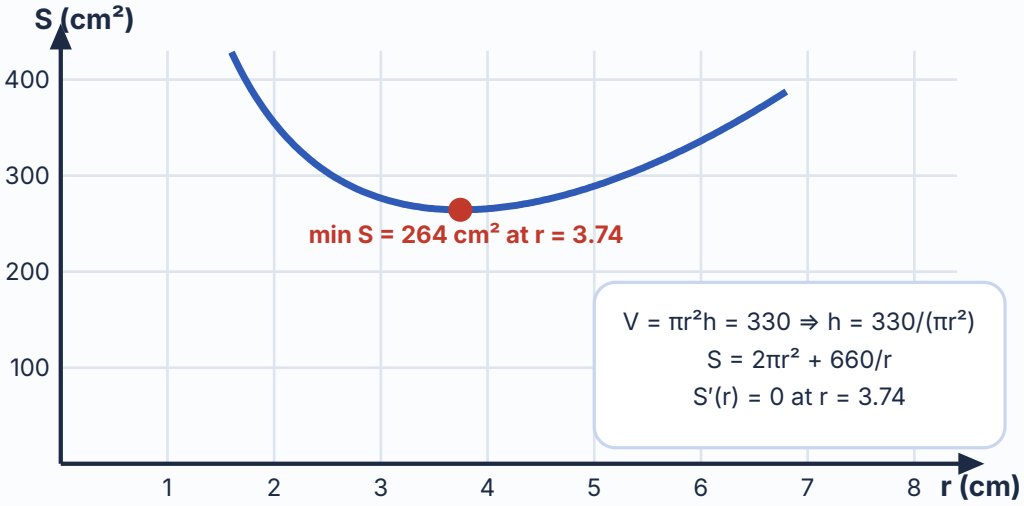
- **Local vs global:** a local maximum is the highest point nearby. On a closed interval, also check the end points: the global maximum might be at an end.

4 Optimisation

WORKED EXAMPLE

A farmer has 100 m of fencing to make a rectangular pen against a long wall (no fence needed along the wall). Find the dimensions for the largest area.

1. two sides y and one side 100 − 2y, so $A = y(100 - 2y) = 100y - 2y^2$ M1 A1
2. $A'(y) = 100 - 4y = 0 \Rightarrow y = 25$ M1
3. **25 m by 50 m, area 1250 m^2** A1



Minimum surface area for a fixed volume: the GDC's minimum tool or S′(r) = 0 gives the same answer.

WORKED EXAMPLE

A closed cylindrical can must hold 330 cm^3. Find the radius that uses the least metal, and that surface area.

1. $h = \frac{330}{\pi r^2}$, so $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{660}{r}$ M1 A1
2. $S'(r) = 4\pi r - \frac{660}{r^2} = 0 \Rightarrow r = 3.74$ cm M1 A1
3. **S_{min} = 264 cm^2 (h = 7.49 cm)** A1

- **Profit models:** $P(x) = -2x^2 + 120x - 1000$ has $P'(x) = -4x + 120 = 0$ at x = 30, so the maximum profit is P(30) = **800**.

EXAM TIP

Always justify that your answer is a maximum (or minimum): a sign change in the derivative, the shape of the graph, or the second derivative (HL). Then answer the question asked, with units.

5 More derivatives and rules HL

f(x)	f′(x)
x^n (any real n)	nx^{n-1}
sin x	cos x
cos x	−sin x
tan x	$\frac{1}{\cos^2 x}$
e^x	e^x
ln x	$\frac{1}{x}$

chain: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$; product: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$; quotient: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du/dx - u\,dv/dx}{v^2}$

FORMULA BOOKLET

the three rules

WORKED EXAMPLE

Differentiate $y = x^2e^{3x}$, $y = \ln(x^2 + 1)$ and $y = \frac{\sin 2x}{x}$.

1. product: $2xe^{3x} + 3x^2e^{3x}$ M1 A1
2. chain: $\frac{2x}{x^2 + 1}$ A1
3. **quotient: $\frac{2x\cos 2x - \sin 2x}{x^2}$** M1 A1

- **Related rates:** link the rates with the chain rule. A balloon's radius grows at 0.5 cm s^{-1}. Since $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$; at r = 10 cm, $\frac{dV}{dt} = 4\pi(100)(0.5) = \mathbf{628\ cm^3\ s^{-1}}$.

Riyaz Sir

6 The second derivative HL

- f″(x) is the rate of change of the gradient. **f″(x) > 0:** concave up (gradient increasing); **f″(x) < 0:** concave down. A **point of inflection** is where the concavity changes.
- **Second derivative test:** at a stationary point, f″ < 0 means a maximum, f″ > 0 a minimum. For f above, f″(x) = 12x − 18: f″(1) = −6 (max), f″(2) = 6 (min); the point of inflection is at x = 1.5.

LINKING QUESTION

Integration reverses differentiation (Integration, areas and kinematics). Optimisation uses the models of Modelling with functions, and HL differential equations are built from derivatives (Differential equations and phase portraits). For more algebraic calculus, see Maths AA Differentiation.

IDEA FOR YOUR IA / EXPLORATION

Optimise something real: the dimensions of a cereal box or drinks can that use the least card or metal for a given volume, and compare your optimum with the real product. Explain why the manufacturer might choose differently.

Test yourself — cover the answers first

QUICK CHECK

1. Differentiate $5x^4$.
2. Find the gradient of $y = x^3$ at $x = 2$.
3. f′(3) = 0 and f′ changes from − to + at 3. What is at x = 3?
4. What is the gradient of the normal if the tangent's gradient is 4?
5. HL Differentiate e^{5x}.
6. HL f″(a) < 0 at a stationary point. Max or min?

EXAM-STYLE

7. The height of a plant is $h(t) = 0.2t^3 - 3t^2 + 15t + 2$ cm, for $0 \le t \le 8$ weeks. Find h′(t), the rate of growth at t = 2, and the time when the growth rate is smallest.
8. An open box has a square base of side x cm and volume 500 cm³. Show that the surface area is $A = x^2 + \frac{2000}{x}$, and find the minimum area.

Show answers

1. $20x^3$
2. 12
3. a local minimum
4. $-\frac{1}{4}$
5. $5e^{5x}$
6. maximum
7. $h'(t) = 0.6t^2 - 6t + 15$
   $h'(2) = 2.4 - 12 + 15 =$ **5.4 cm per week**
   h' is smallest at its vertex $t = \frac{6}{1.2} =$ **5 weeks** ($h'(5) = 0$)
8. height $= \frac{500}{x^2}$, so $A = x^2 + 4x \times \frac{500}{x^2} = x^2 + \frac{2000}{x}$ AG
   $A' = 2x - \frac{2000}{x^2} = 0 \Rightarrow x = 10$
   $A_{min} = 300$ cm²

Stuck on a question?

Send it to me on WhatsApp. I read every message.

Want regular help? Send this to your parent →

Share with a friend

Ask Riyaz Sir