IB DP MATHS AI · TOPIC 3 GEOMETRY AND TRIGONOMETRY · SL 3.1–3.6 · ~8 MIN READ

Trigonometry, 3D geometry and Voronoi diagrams

3D distance, midpoint, volume and surface area, right-angled trigonometry with angles of elevation and depression, the sine and cosine rules, area ½ab sin C, bearings, arcs and sectors in degrees, perpendicular bisectors, and Voronoi diagrams with the toxic waste dump problem.

3D distance & midpoint | Volume & surface area | Angles of elevation & depression | Sine rule | Cosine rule | Area ½ab sin C | Bearings | Arcs & sectors

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No tag = **SL and HL** HL ONLY HL students only FORMULA BOOKLET given in the exam

M1 A1 how marks are awarded highlight = comes up again and again Test yourself at the end

IN 30 SECONDS

- **Right-angled triangles:** SOH CAH TOA. **Any triangle:** the sine rule when you know a side and its opposite angle; the cosine rule for two sides and the angle between them, or for three sides.
- **Bearings** are measured clockwise from North, with three figures (040°). Draw a North line at every point.
- **Arcs and sectors (degrees):** arc length = $\frac{\theta}{360} \times 2\pi r$ and sector area = $\frac{\theta}{360} \times \pi r^2$.
- A **Voronoi diagram** splits a region into cells, one per **site**: every point in a cell is closer to that site than to any other. Cell edges lie on **perpendicular bisectors**.
- This note is the same at SL and HL, and unchanged for 2029. (In AI SL all trigonometry is in degrees; radians are HL.)

BY THE END YOU CAN

- Find distances and midpoints in 3D, and volumes and surface areas of solids.
- Use right-angled trigonometry, including angles of elevation and depression, and angles in 3D.
- Use the sine rule, the cosine rule and ½ab sin C, including bearings problems.
- Find arc lengths and sector areas with angles in degrees.
- Find equations of perpendicular bisectors.
- Construct and interpret Voronoi diagrams, including nearest neighbour interpolation and the largest empty circle.

1 3D geometry

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$ FORMULA BOOKLET

distance in 3D

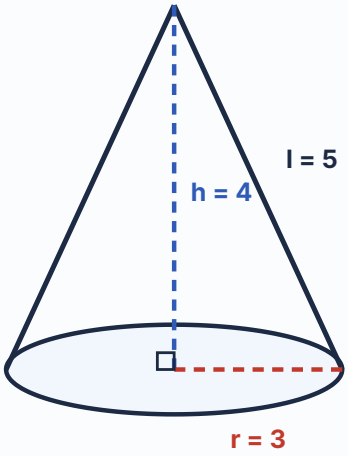
$$\text{midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$ FORMULA BOOKLET

average each coordinate

WORKED EXAMPLE

Find the distance between (1, 2, 3) and (4, 6, 15).

1. $\sqrt{(3^2 + 4^2 + 12^2)} = \sqrt{169}$ M1
2. **= 13** A1



slant height $l = \sqrt{(r^2 + h^2)} = 5$

$V = \frac{1}{3}\pi r^2 h = 12\pi = 37.7$

curved area $= \pi r l = 15\pi$

total SA $= \pi r l + \pi r^2$

$= 24\pi = 75.4$

For cones, the slant height l comes from Pythagoras with r and h.

Solid	Volume	Surface area
Cuboid	lwh	$2(lw + lh + wh)$
Cylinder	$\pi r^2 h$	$2\pi rh + 2\pi r^2$
Cone	$\frac{1}{3}\pi r^2 h$	$\pi rl + \pi r^2$
Sphere	$\frac{4}{3}\pi r^3$	$4\pi r^2$
Pyramid	$\frac{1}{3} \times$ base area $\times h$	base + triangular faces

WORKED EXAMPLE

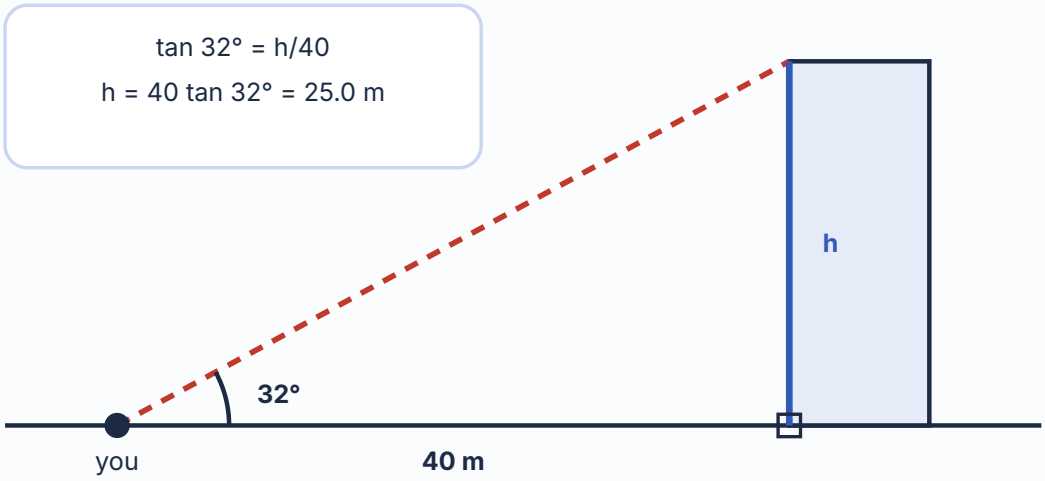
A cone has radius 3 cm and height 4 cm. Find its volume and total surface area.

1. $V = \frac{1}{3}\pi(9)(4) = 12\pi =$ **37.7 cm^3** A1
2. $l = \sqrt{(9 + 16)} = 5$ A1
3. **$SA = \pi(3)(5) + \pi(9) = 24\pi = 75.4$ cm^2** M1 A1

COMMON MISTAKE

Using the vertical height h in the curved area πrl. The curved surface needs the **slant** height l.

2 Right-angled trigonometry



Angles of elevation are measured up from the horizontal; angles of depression down from the horizontal.

WORKED EXAMPLE

From 40 m away, the angle of elevation to the top of a building is 32°. Find its height.

1. $\tan 32° = \frac{h}{40}$ M1
2. **h = 25.0 m** A1

- **3D angles:** to find the angle between a line and a plane, find the right-angled triangle that contains the line and its "shadow" on the plane, then use SOH CAH TOA.

3 Sine rule, cosine rule and area

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ FORMULA BOOKLET

sine rule

$$a^2 = b^2 + c^2 - 2bc\cos A,\ \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$ FORMULA BOOKLET

cosine rule

$$\text{area} = \tfrac{1}{2}ab\sin C$$ FORMULA BOOKLET

two sides and the angle between them

WORKED EXAMPLE

In triangle ABC, A = 48°, B = 70° and a = 12 m. Find b.

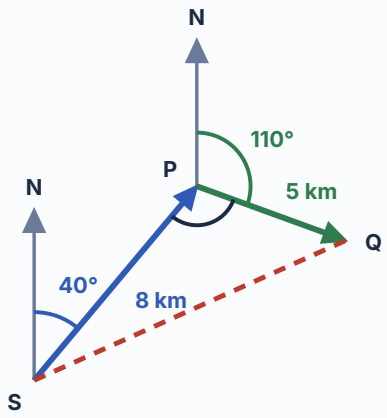
1. $\frac{b}{\sin 70°} = \frac{12}{\sin 48°}$ M1
2. **b = 15.2 m** A1

WORKED EXAMPLE

A field is a triangle with sides 50 m and 70 m and an angle of 65° between them. Find its area.

1. ½ × 50 × 70 × sin 65° M1
2. **= 1590 m² (1586)** A1

4 Bearings



angle SPQ = 180° − 70°
= 110°
SQ² = 8² + 5² − 2(8)(5)cos 110°
SQ = 10.8 km

Use the North lines to find the angle inside the triangle, then the cosine rule.

WORKED EXAMPLE

A boat sails 8 km from S on a bearing of 040° to P, then 5 km on a bearing of 110° to Q. Find the distance SQ and the bearing of Q from S.

1. angle SPQ = 180° − (110° − 40°) = 110° (co-interior angles) M1 A1
2. $SQ^2 = 64 + 25 - 80\cos 110° \Rightarrow$ **SQ = 10.8 km** M1 A1
3. **sine rule: angle PSQ = 25.8°, so the bearing is 40° + 25.8° = 066°** M1 A1

EXAM TIP

An angle of depression from A to B equals the angle of elevation from B to A (alternate angles). Sketch it with a horizontal line at the top point.

5 Arcs and sectors

$$l = \frac{\theta}{360} \times 2\pi r,\ A = \frac{\theta}{360} \times \pi r^2$$ FORMULA BOOKLET

θ in degrees

WORKED EXAMPLE

A sector has radius 6 cm and angle 75°. Find its arc length and area.

1. $l = \frac{75}{360} \times 2\pi \times 6 =$ **7.85 cm** A1
2. **$A = \frac{75}{360} \times \pi \times 36 = 23.6\ cm^2$** A1

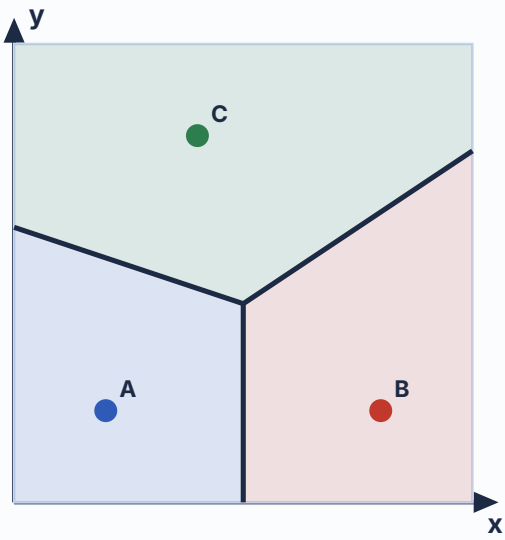
6 Perpendicular bisectors and Voronoi diagrams

- **Perpendicular bisector of AB:** passes through the midpoint of AB with gradient $-1/m_{AB}$. Every point on it is the same distance from A and from B.

WORKED EXAMPLE

Find the perpendicular bisector of A(2, 2) and C(4, 8).

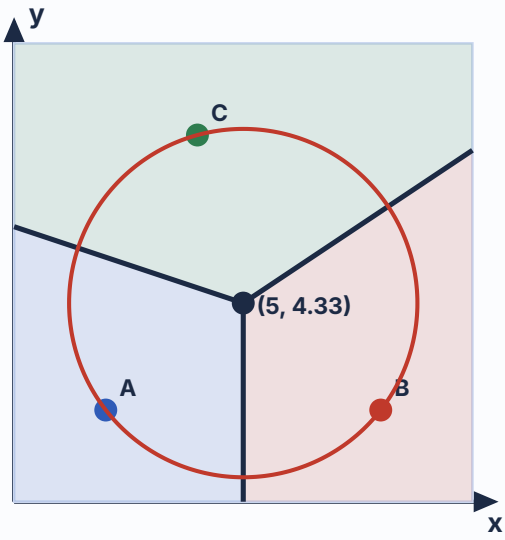
1. midpoint (3, 5); gradient of AC = 3, so the perpendicular gradient is −⅓ M1 A1
2. **y − 5 = −⅓(x − 3), so y = −⅓x + 6** A1



sites A(2, 2), B(8, 2), C(4, 8)
each cell: points closest to its site
edges lie on perpendicular bisectors of pairs of sites

Three sites and their cells. The three edges meet at one vertex.

- **Vocabulary:** sites (the given points), cells (regions), edges (cell boundaries, on perpendicular bisectors) and vertices (where edges meet).
- **Nearest neighbour interpolation:** estimate a value at a new point using the value at the site of the cell it lies in (for example, rainfall measured at the nearest weather station).
- **Adding a site:** draw the perpendicular bisectors between the new site and its neighbours, then remove the parts of old edges that are now inside the new cell.



the vertex (5, 4.33) is equidistant from A, B and C
largest empty circle: radius 3.80
(the "toxic waste dump" point)

The point furthest from every site (inside the region) is at a Voronoi vertex or on the boundary.

WORKED EXAMPLE

Find the coordinates of the vertex where the cells of A(2, 2), B(8, 2) and C(4, 8) meet, and its distance from each site.

1. the bisector of AB is x = 5 **A1**
2. substitute in y = −⅓x + 6 (bisector of AC): y = $\frac{13}{3}$ **M1**
3. **vertex (5, 4.33), distance √(3² + 2.33²) = 3.80** **A1**

- **Toxic waste dump problem:** to place something as far as possible from all sites (towns), test the Voronoi vertices and the corners and edges of the region; the best point is the centre of the largest empty circle.

COMMON MISTAKE

Drawing cell edges as lines joining the sites. Edges are **perpendicular bisectors** of those joining lines, and they stop where they meet other edges.

LINKING QUESTION

Bearings use the same trig as vectors in Radians, vectors and matrix transformations (HL). Voronoi diagrams link to Graph theory and to real geography: the nearest hospital, school or phone mast. For trigonometry in Maths AA with radians and identities, see Trigonometry: triangles, radians, arcs and sectors.

IDEA FOR YOUR IA / EXPLORATION

Build a Voronoi diagram for real locations in your city (metro stations, hospitals or coffee shops) from a map, then use it to find the area each one serves or the best place for a new one.

Test yourself — cover the answers first

QUICK CHECK

1. Find the midpoint of (2, −1, 4) and (6, 3, 0).
2. Which rule finds an angle from three sides?
3. What is the bearing of due West?
4. Find the arc length for r = 10 cm and θ = 36°.
5. On which lines do Voronoi edges lie?

EXAM-STYLE

6. A tower stands on flat ground. From point P the angle of elevation of the top is 25°; from Q, 30 m closer, it is 40°. Find the height of the tower.
7. Rainfall is recorded at the sites A(2, 2): 20 mm, B(8, 2): 35 mm and C(4, 8): 28 mm. Use the Voronoi diagram to estimate the rainfall at (7, 4).

Show answers

1. (4, 1, 2)
2. the cosine rule
3. 270°
4. $2\pi \approx 6.28$ cm
5. perpendicular bisectors of pairs of sites
6. in triangle PQT: angle PTQ = 40° − 25° = 15°, so QT = $\frac{30 \sin 25°}{\sin 15°}$ = 48.99 m

   h = QT sin 40°

   h = 31.5 m
7. (7, 4) is right of x = 5 and below the B–C edge (y = 5 + ⅔(7 − 6) = 5.67 at x = 7)

   so it is in B's cell

   estimate 35 mm

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