IGCSE MATHEMATICS 0580 · CHAPTER 11 NOTES · ~21 MIN READ

Probability

From the probability scale to tree diagrams and conditional probability. Every method is drawn with sample space tables, tree diagrams and Venn diagrams, so you can see why you multiply for AND and add for OR.

Probability scale | Relative & expected frequency | Combined events | Tree diagrams | Conditional probability

RS by Riyaz Sir
mentorriyaz.in

No tag = **Core** (everyone) EXTENDED Extended only highlight = comes up again and again

Test yourself at the end

IN 30 SECONDS

- **Probability** = favourable outcomes ÷ total outcomes. It is always between **0** and **1**, and **P(not A) = 1 − P(A)**. Give it as a fraction, decimal or percentage.
- **Relative frequency** = successes ÷ trials, and is used when you **cannot** work out the probability. **Expected frequency = probability × number of trials.**
- **Combined events:** list outcomes in a **table** or **tree diagram**. For **AND multiply** along the branches. For **OR add** the outcomes. Without replacement the fractions **change**. EXT

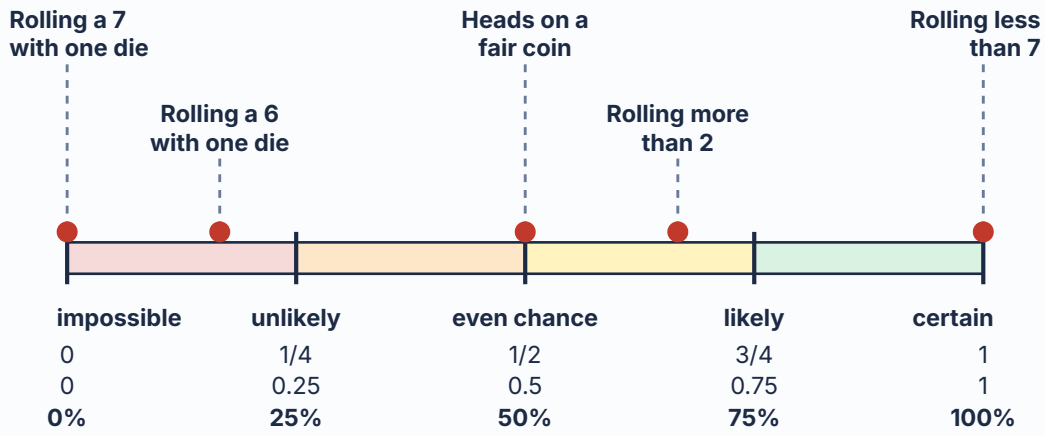
BY THE END YOU CAN

- Use the probability scale and calculate the probability of a single event.
- Find the probability that an event does not happen.
- Use relative frequency to estimate probability and find expected frequencies.
- Use sample space tables and tree diagrams for combined events.
- Use the rules P(A and B) = P(A) × P(B) and P(A or B) = P(A) + P(B) correctly.
- Solve problems without replacement, use Venn diagrams and find conditional probability. EXT

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1 Probability basics

The probability scale



Probability is always between 0 and 1. Fractions, decimals and percentages all work.
Probabilities of all the outcomes add up to 1.

Every probability sits on a line from 0 (impossible) to 1 (certain). The middle is an even chance, 1/2.

Probability — a number from **0 to 1** that measures **how likely** an event is. **0** means it cannot happen and **1** means it is certain.

Outcome and event — an **outcome** is one possible result, such as rolling a 4. An **event** is one or more outcomes, such as "an even number".

Fair — every outcome is **equally likely**, for example a fair coin or an unbiased die.

- **Write probabilities as fractions, decimals or percentages.** Never write them as ratios such as 1 : 5, and never write more than 1 or less than 0.
- **Simplify fractions.** 2/6 should be written as 1/3.
- **Words are not enough.** "Likely" scores no marks if you are asked for a probability. Give a number.

WORKED EXAMPLE

A fair die is rolled. Find the probability of rolling a number less than 3.

1. favourable outcomes: 1 and 2, so 2
2. total outcomes: 6
3. **P = 2/6 = 1/3**

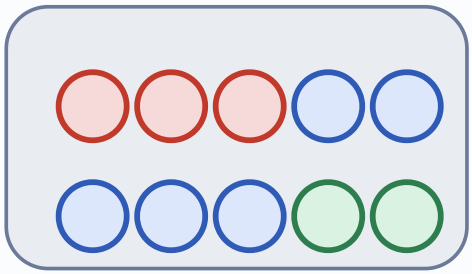
COMMON MISTAKE

Writing a probability as **greater than 1**, or as a ratio. A probability must be between 0 and 1.

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Calculating a probability

A bag of 10 counters



3 red, 5 blue, 2 green
one counter is picked at random

P(event) = favourable ÷ total

P(red)	$\frac{3}{10}$	= 0.3
P(blue)	$\frac{5}{10}$	= 1/2
P(green)	$\frac{2}{10}$	= 1/5

P(not red) = 1 − P(red)
= 1 − 3/10 = 7/10

All three probabilities add up to 1: 3/10 + 5/10 + 2/10 = 1

Count the outcomes you want, divide by the total number of outcomes, then simplify.

$$P(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of equally likely outcomes}}$$

the basic formula

$$P(\text{not A}) = 1 - P(\text{A})$$

the probability that A does not happen

- **Check that all outcomes are equally likely** before using the formula. A fair spinner with unequal sectors is **not**.
- **All the probabilities in an experiment add up to 1.** This is a quick check on your answers.
- **OR with one pick:** for a single pick, add the probabilities of the separate outcomes, for example P(red or blue) = P(red) + P(blue).

WORKED EXAMPLE

A bag has 3 red, 5 blue and 2 green counters. Find P(green), P(not red) and P(red or blue).

1. total = 3 + 5 + 2 = 10
2. P(green) = 2/10 = **1/5**
3. P(not red) = 1 − 3/10 = **7/10**
4. **P(red or blue) = 3/10 + 5/10 = 8/10 = 4/5**

WORKED EXAMPLE

A fair spinner is numbered 1 to 8. Find the probability of getting (a) a multiple of 3 (b) a prime number.

1. (a) 3 and 6, so 2/8 = **1/4**
2. **(b) 2, 3, 5 and 7, so 4/8 = 1/2**

WORKED EXAMPLE

The probability that it rains tomorrow is 0.35. Find the probability that it does not rain.

1. **1 − 0.35 = 0.65**

WORKED EXAMPLE

A letter is picked at random from the word MATHEMATICS. Find P(M) and P(vowel).

1. there are 11 letters in total
2. M appears 2 times, so P(M) = **2/11**
3. **vowels are A, E, A and I, so P(vowel) = 4/11**

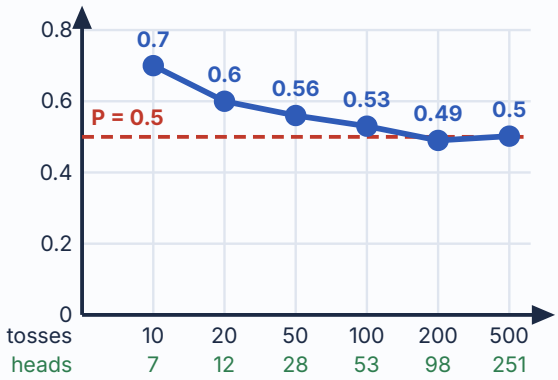
EXAM TIP

"Not" means 1 minus. It is often quicker to work out the probability of the opposite event.

2 Relative and expected frequency

Estimating from experiments

Relative frequency of heads



Relative frequency

= successes ÷ trials
With more trials it settles close to the true probability.

Expected frequency
= P × number of trials
P(6) = 1/6, 300 rolls:
300 × 1/6 = 50

The more you toss, the closer the relative frequency gets to the probability.

The relative frequency settles down towards the true probability as the number of trials increases. Expected frequency works the other way round.

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Relative frequency — the **fraction of trials** in which an event happened: **number of successes ÷ total number of trials**. It is an **estimate** of the probability.

Expected frequency — how many times you **expect** an event to happen: **probability × number of trials**.

- **More trials means a better estimate.** 10 throws tell you little, but 1000 throws are much more reliable.
- **When there is no theory to work out the probability** (a drawing pin, a biased die) use the relative frequency.
- **Expected does not mean exact.** If you expect 50 sixes you will probably get a number close to 50, not exactly 50.
- **Is it fair?** If the results are very different from what you expect, and there are many trials, the die or coin may be **biased**.

WORKED EXAMPLE

A drawing pin lands point up 36 times in 120 throws. Estimate the probability that it lands point up, and the number of times you expect it to land point up in 500 throws.

1. relative frequency = 36 ÷ 120 = **0.3**
2. **expected = 0.3 × 500 = 150**

WORKED EXAMPLE

A fair die is rolled 300 times. How many times do you expect (a) a 6 (b) a multiple of 3?

1. (a) 1/6 × 300 = **50**
2. **(b) 2/6 × 300 = 100**

WORKED EXAMPLE

A coin is tossed 200 times and lands on heads 120 times. Is there evidence that the coin is biased?

1. relative frequency of heads = 120 ÷ 200 = 0.6
2. for a fair coin we expect 0.5, which is 100 heads
3. **120 is quite far from 100, so the coin may be biased, but a larger number of trials would give more reliable evidence**

COMMON MISTAKE

Thinking that **past results change the next one**. After five heads in a row the probability of heads on the next toss is still **1/2**.

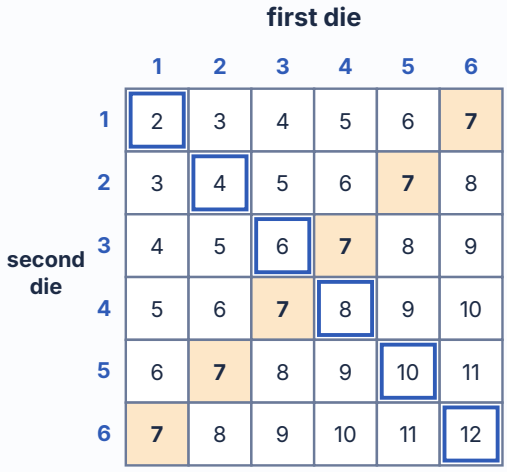
EXAM TIP

If the question says **"estimate"** use the **relative frequency**. If it says **"expect"** use **probability × number of trials**.

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3 Combined events

Sample space diagrams



Two dice: 36 outcomes

each cell shows the total

total is 7

a double

P(total = 7) = 6/36 = 1/6
P(a double) = 6/36 = 1/6
P(total ≥ 10) = 6/36 = 1/6
P(total = 12) = 1/36
P(total = 1) = 0

List all outcomes in a table, then count the ones you want.

A table lists every outcome of two events. Count the cells you want, then divide by the total number of cells.

Sample space — the **list or table of all possible outcomes**. For two dice there are 6 × 6 = **36** outcomes.

- **Make sure outcomes are equally likely** and that you have not missed any. Check that your table has the right number of cells.
- **Shade or circle** the cells you want, then count them.
- **For two spinners or a coin and a die,** make a table or list. Coin and die gives 2 × 6 = 12 outcomes.

WORKED EXAMPLE

Two fair dice are rolled. Find the probability that (a) the total is 7 (b) the two numbers are different (c) the product is 12.

1. total outcomes = 36
2. (a) 6 cells have a total of 7: **6/36 = 1/6**
3. (b) 6 doubles, so 30 are different: **30/36 = 5/6**
4. **(c) (2, 6), (3, 4), (4, 3) and (6, 2), so 4/36 = 1/9**

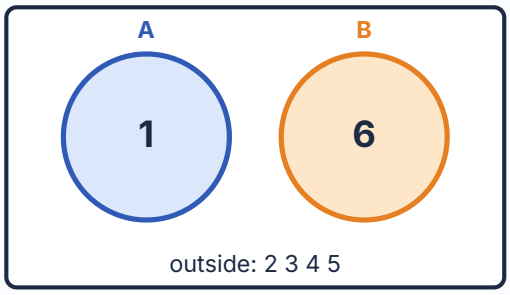
WORKED EXAMPLE

A fair coin is tossed and a fair die is rolled. Find the probability of a head and an even number.

1. outcomes: H2, H4 and H6 are favourable out of 12
2. **3/12 = 1/4**

AND and OR rules

Mutually exclusive

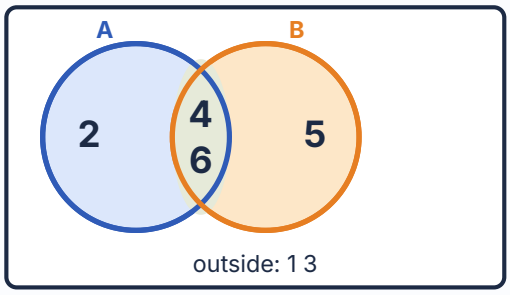


No overlap: they cannot both happen.

P(A or B) = P(A) + P(B)

A = roll a 1, B = roll a 6
1/6 + 1/6 = 1/3

Not mutually exclusive



They overlap: remove the overlap once.

P(A or B) = P(A) + P(B) − P(A and B)

A = even, B = more than 3
3/6 + 3/6 − 2/6 = 4/6 = 2/3

Add the probabilities for OR. Multiply them for AND (independent events).

AND means both happen, so multiply. OR means either can happen, so add. Check the conditions first.

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Independent events — one event has **no effect** on the other, such as two separate coin tosses or two dice.

Mutually exclusive — two events that **cannot both happen** at the same time, such as rolling a 1 and rolling a 6 on one die.

$$P(A \text{ and } B) = P(A) \times P(B)$$

A and B independent

$$P(A \text{ or } B) = P(A) + P(B)$$

A and B mutually exclusive

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

any two events EXT

- **"At least one"** is easiest by the opposite: P(at least one) = **1 − P(none)**.
- **If the events overlap,** adding the two probabilities counts the overlap twice. Take one copy away. EXT

WORKED EXAMPLE

The probability of rain on Monday is 0.3 and on Tuesday is 0.4. The days are independent. Find P(rain on both days) and P(rain on at least one day).

1. both: 0.3 × 0.4 = **0.12**
2. P(no rain Monday) = 0.7 and P(no rain Tuesday) = 0.6
3. P(no rain at all) = 0.7 × 0.6 = 0.42
4. **P(at least one) = 1 − 0.42 = 0.58**

WORKED EXAMPLE

A fair die is rolled. Find P(1 or 6).

1. a die cannot show both, so the events are mutually exclusive
2. **1/6 + 1/6 = 1/3**

WORKED EXAMPLE

A fair die is rolled. A is "even" and B is "more than 3". Find P(A or B).

1. P(A) = 3/6 and P(B) = 3/6
2. A and B both: 4 and 6, so P(A and B) = 2/6
3. **P(A or B) = 3/6 + 3/6 − 2/6 = 4/6 = 2/3** EXT

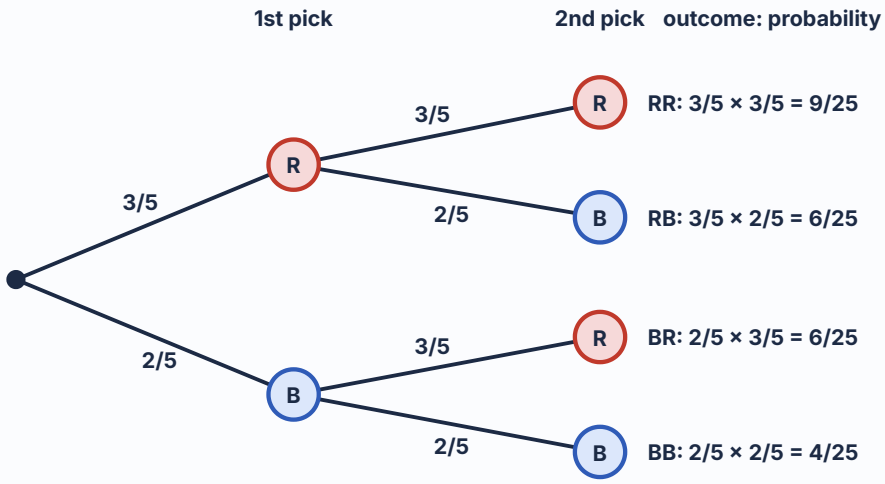
COMMON MISTAKE

Adding the probabilities for AND, or **multiplying** them for OR. Remember: AND is times, OR is plus.

COMMON MISTAKE

Using the AND rule when the events are **not independent**. This happens when the first event changes the second.

Tree diagrams with replacement



With replacement the branches repeat. 9/25 + 6/25 + 6/25 + 4/25 = 1

Each pair of branches adds to 1. Multiply along a path for AND, then add the paths you want for OR.

- **Draw the branches** for the first event, then repeat for the second. Write the **probability on each branch** and the **outcome at the end**.
- **Branches from one point add up to 1.** For this bag: 3/5 + 2/5 = 1.
- **Along the branches multiply.** Between different outcomes add.
- **With replacement** the counter is put back, so the second set of branches is the **same** as the first.

WORKED EXAMPLE

A bag has 3 red and 2 blue counters. A counter is picked, replaced, and a second is picked. Find P(both red), P(one of each colour) and P(at least one blue).

1. P(R) = 3/5 and P(B) = 2/5 each time
2. P(RR) = 3/5 × 3/5 = **9/25**
3. one of each: RB + BR = 6/25 + 6/25 = **12/25**
4. **at least one blue = 1 − P(RR) = 1 − 9/25 = 16/25**

WORKED EXAMPLE

Find the probability that both counters are the same colour.

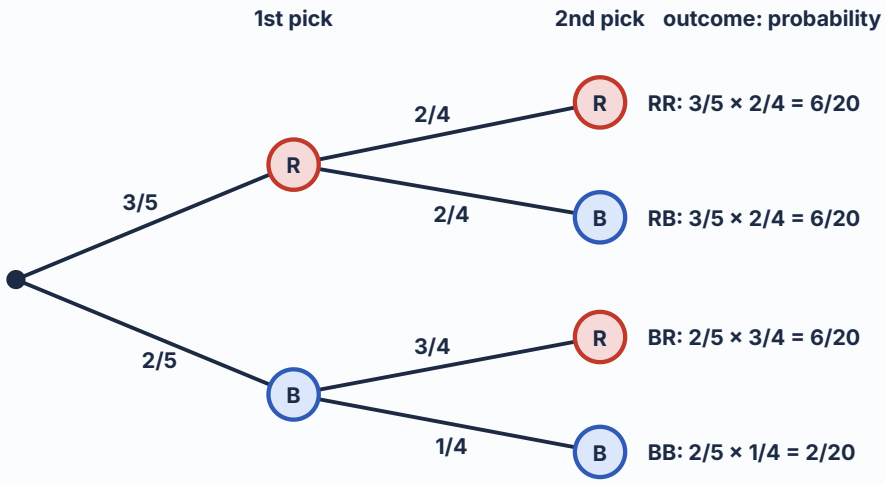
1. **P(RR) + P(BB) = 9/25 + 4/25 = 13/25**

COMMON MISTAKE

Not checking that the **final outcomes add up to 1**. If they do not, one of your fractions is wrong.

Without replacement

EXTENDED



Without replacement the second branches change. 6/20 + 6/20 + 6/20 + 2/20 = 1

When the first counter is not put back, there is one fewer counter in the bag, so the second set of branches changes.

Dependent events — events where the **first outcome changes the probabilities** of the second.

- **Reduce the total by 1** on the second pick, and reduce the number of the colour you took out.
- **The second branches still add up to 1.** For example, after taking a red: 2/4 + 2/4 = 1.
- **The method is the same:** multiply along the branches, and add the paths you want.

WORKED EXAMPLE

A bag has 3 red and 2 blue counters. Two counters are picked without replacement. Find P(both red), P(different colours) and P(both blue).

1. P(RR) = 3/5 × 2/4 = 6/20 = **3/10**
2. P(RB) = 3/5 × 2/4 = 6/20 and P(BR) = 2/5 × 3/4 = 6/20
3. different colours = 12/20 = **3/5**
4. **P(BB) = 2/5 × 1/4 = 2/20 = 1/10**

WORKED EXAMPLE

A bag has 5 red and 3 blue counters. Two are picked without replacement. Find P(both blue).

1. **3/8 × 2/7 = 6/56 = 3/28**

COMMON MISTAKE

Using the **same fractions** on both pick levels in a "without replacement" question. After the first pick the totals change.

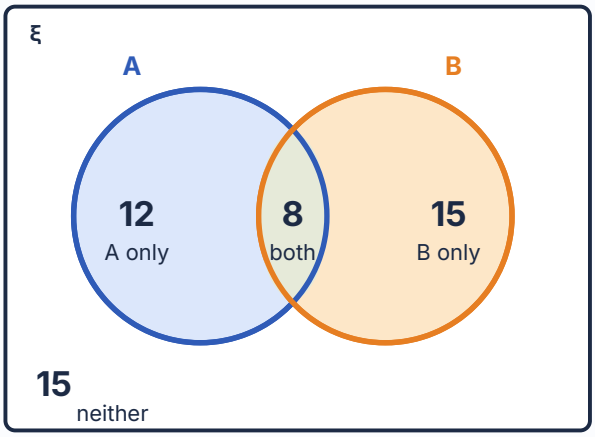
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Venn diagrams and probability

EXTENDED

Survey of 50 students

A = plays football, B = plays tennis



P(A) = 20/50 = 2/5

P(B) = 23/50

P(A and B) = 8/50 = 4/25

P(A or B) = 35/50 = 7/10

P(neither) = 15/50 = 3/10

P(A given B) = 8/23
only look inside circle B:
23 students, 8 are in A

Each region of a Venn diagram holds a count. Probability = the numbers you want ÷ the total for the whole diagram.

- **Fill in the middle first:** the number in **both** sets. Then work outwards to the "only" regions and the number in **neither**.
- **Total in A** = A only + both. **A or B** means all of A and B together, which is everything except the "neither" region.
- **Check:** all the numbers in the diagram must add up to the total.

WORKED EXAMPLE

50 students were asked about football (A) and tennis (B). 20 play football, 23 play tennis and 8 play both. Find P(neither).

1. A only = 20 − 8 = 12
2. B only = 23 − 8 = 15
3. at least one = 12 + 8 + 15 = 35
4. neither = 50 − 35 = 15
5. **P(neither) = 15/50 = 3/10**

WORKED EXAMPLE

In a class of 30, 18 study French, 12 study Spanish and 5 study both. Find P(neither).

1. French only = 18 − 5 = 13, Spanish only = 12 − 5 = 7
2. at least one = 13 + 5 + 7 = 25
3. neither = 30 − 25 = 5
4. **P(neither) = 5/30 = 1/6**

Conditional probability

EXTENDED

60 students: how they travel to school

	Walk	Bus	Total
Boys	14	10	**24**
Girls	16	20	**36**
Total	**30**	**30**	**60**

given that they walk:
only look at the walk column

P(walk) = 30/60 = 1/2

P(girl and walk) = 16/60 = 4/15

P(girl given walk) = 16 ÷ 30 = 8/15

P(walk given girl) = 16 ÷ 36 = 4/9

The word "given" tells you which total to divide by.

The word "given" tells you the condition. Look only at the people or outcomes that meet the condition, then divide by that total.

Conditional probability — the probability that A happens **given that** B has already happened. It is written **P(A | B)**.

$$P(A \text{ given } B) = \frac{\text{number in both A and B}}{\text{number in B}}$$

restrict to B, then count

$$P(A \text{ given } B) = \frac{P(A \text{ and } B)}{P(B)}$$

the same rule using probabilities

- **The condition becomes the new total.** If you are told "given that the student walks", the total is the number who walk.
- **P(A given B) is not the same as P(B given A).** The first divides by the total for B, the second by the total for A.
- **With a tree diagram:** the second branches already show conditional probabilities. After a red counter, the probability of another red is 2/4.

WORKED EXAMPLE

60 students were asked how they travel: boys 14 walk and 10 take the bus; girls 16 walk and 20 take the bus. Find P(girl given walks) and P(walks given girl).

1. total who walk = 14 + 16 = 30
2. P(girl given walks) = 16 ÷ 30 = **8/15**
3. total girls = 16 + 20 = 36
4. **P(walks given girl) = 16 ÷ 36 = 4/9**

WORKED EXAMPLE

Using the Venn diagram above, find P(football given tennis).

1. 23 students play tennis, and 8 of them also play football
2. **8/23**

WORKED EXAMPLE

Two counters are picked without replacement from a bag of 3 red and 2 blue. Given that the first is red, find the probability that the second is red.

1. 2 red and 2 blue are left, so 2 of 4
2. **1/2**

COMMON MISTAKE

Dividing by the **wrong total**. For "given B" divide by the number in **B**, not by everyone.

Test yourself — cover the answers first

QUICK CHECK

1. A fair die is rolled. Find the probability of rolling a prime number.
2. A fair die is rolled. Find the probability of rolling a number greater than 4.
3. The probability that it rains is 0.35. Find the probability that it does not rain.
4. A bag has 4 red and 6 blue counters. One is picked at random. Find P(blue).
5. A fair spinner is numbered 1 to 5. Find the probability of an odd number.
6. The probability that it rains on a day is 0.2. How many rainy days do you expect in 30 days?
7. A coin is tossed 80 times and lands on heads 50 times. Find the relative frequency of heads.
8. A fair die is rolled 120 times. How many 5s do you expect?
9. Two independent events have probabilities 0.4 and 0.5. Find P(both happen).
10. Two mutually exclusive events have probabilities 0.3 and 0.45. Find P(A or B).
11. Two fair coins are tossed. Find the probability of two heads.
12. Two fair dice are rolled. Find the probability of a double six.
13. Two fair dice are rolled. Find the probability that the total is 4.
14. What do the probabilities on a set of branches from one point add up to?
15. A bag has 2 red and 3 blue counters. A counter is picked, replaced and another is picked. Find P(red then blue).
16. EXT The same bag is used, but the first counter is not replaced. Find P(red then blue).
17. EXT The same bag, no replacement. Find P(both red).
18. EXT P(A) = 0.5, P(B) = 0.3 and P(A and B) = 0.1. Find P(A or B).
19. EXT A Venn diagram of 40 students shows 15 in A only, 10 in both, 8 in B only and 7 in neither. Find P(A).
20. EXT In the same Venn diagram, find P(B given A).
21. EXT Write down the formula for P(A given B) using P(A and B) and P(B).

EXAM-STYLE

22. A bag has 5 red, 3 blue and 2 green counters. One counter is picked at random. Find P(red), P(not green), and the number of blue counters you expect in 200 picks with replacement. [4]
23. A spinner is spun 80 times. It lands on red 30 times, blue 30 times and green 20 times. Find the relative frequency of green and estimate the number of greens in 500 spins. [3]
24. Two fair dice are rolled. Find the probability that the total is 9 and the probability that the two numbers are different. [4]
25. Ali scores a penalty with probability 0.7. He takes two penalties. Find the probability that he scores exactly once. [3]
26. A bag has 4 red and 6 blue counters. A counter is picked, replaced and then another is picked. Find the probability that both counters are the same colour. [4]
27. EXT A bag has 4 red and 6 blue counters. Two counters are picked without replacement. Find the probability that both are the same colour. [4]
28. EXT For the same bag and the same picks, find the probability that at least one counter is red. [3]
29. EXT In a class of 30 students, 18 study French, 12 study Spanish and 5 study both. Find (a) the number who study neither, (b) P(neither), (c) P(French given Spanish). [5]
30. EXT Of 80 students, 45 are girls. 20 girls and 15 boys wear glasses. Find P(wears glasses) and P(girl given wears glasses). [3]
31. EXT P(A) = 0.6, P(B) = 0.5 and P(A or B) = 0.8. Find P(A and B) and decide if A and B are independent. [3]

32. EXT A bag has 5 red and 3 blue counters. Three counters are picked without replacement. Find the probability that all three are red. [3]

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Show answers

1. 1/2 (2, 3 and 5)
2. 1/3
3. 0.65
4. 3/5
5. 3/5
6. 6
7. 5/8 (0.625)
8. 20
9. 0.2
10. 0.75
11. 1/4
12. 1/36
13. 1/12
14. 1
15. 6/25
16. 3/10
17. 1/10
18. 0.7
19. 5/8
20. 2/5
21. P(A and B) ÷ P(B)
22. P(red) = 5/10 = 1/2
    P(not green) = 8/10 = 4/5
    expected blue = 3/10 × 200 = 60
23. relative frequency = 20 ÷ 80 = 0.25
    expected = 0.25 × 500 = 125
24. total 9: (3, 6), (4, 5), (5, 4), (6, 3) so 4/36 = 1/9
    doubles: 6, so different = 30/36 = 5/6
25. score then miss: 0.7 × 0.3 = 0.21
    miss then score: 0.3 × 0.7 = 0.21
    total = 0.42
26. P(RR) = 0.4 × 0.4 = 0.16
    P(BB) = 0.6 × 0.6 = 0.36
    total = 0.52
27. P(RR) = 4/10 × 3/9 = 12/90
    P(BB) = 6/10 × 5/9 = 30/90
    total = 42/90 = 7/15
28. at least one red = 1 − P(both blue)
    P(BB) = 30/90
    1 − 30/90 = 60/90 = 2/3

29. French only 13, Spanish only 7, both 5
    (a) 30 − 25 = 5
    (b) 5/30 = 1/6
    (c) 5 ÷ 12 = 5/12

30. glasses = 20 + 15 = 35
    P(glasses) = 35/80 = 7/16
    P(girl given glasses) = 20/35 = 4/7

31. P(A and B) = 0.6 + 0.5 − 0.8 = 0.3
    P(A) × P(B) = 0.6 × 0.5 = 0.3
    they are equal, so A and B are independent

32. 5/8 × 4/7 × 3/6 = 60/336
    = 5/28

Stuck on a question?

Send it to me on WhatsApp. I read every message.

Want regular help? Send this to your parent →

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