IGCSE MATHEMATICS 0580 · CHAPTER 9 NOTES · ~34 MIN READ

# Pythagoras & trigonometry

Right-angled triangles first (Pythagoras, SOH CAH TOA, bearings), then the sine rule, cosine rule, 3D problems and trig graphs. Every method is shown on a labelled diagram, so you can see which rule to pick.

Pythagoras | SOHCAHTOA | Bearings | Angles of elevation | Sine & cosine rules | Area ½ab sin C | 3D | Trig graphs

RS by Riyaz Sir
mentorriyaz.in

No tag = **Core** (everyone) EXTENDED Extended only highlight = comes up again and again

Test yourself at the end

## IN 30 SECONDS

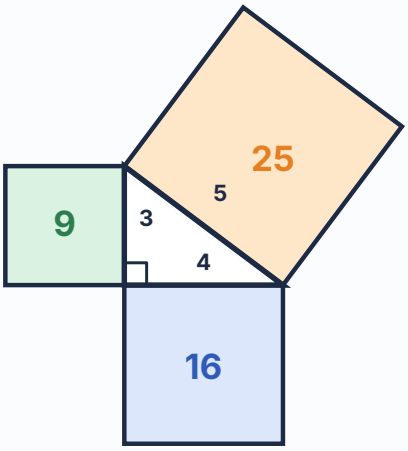
- **Pythagoras:** in a right-angled triangle $c^2 = a^2 + b^2$. **Add** the squares to find the hypotenuse and **subtract** to find a shorter side.
- **SOH CAH TOA** links an angle to two sides. **Label** the sides (opposite, adjacent, hypotenuse), pick the ratio, then solve. Use $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ to find angles.
- **Not right-angled?** Use the **sine rule** for a side and its opposite angle (a pair), and the **cosine rule** for two sides and the angle between them, or three sides. EXT

## BY THE END YOU CAN

- Use Pythagoras' theorem to find a missing side and to test for a right angle.
- Use sine, cosine and tangent to find sides and angles in right-angled triangles.
- Solve problems with three-figure bearings, and angles of elevation and depression.
- Use the sine rule, the cosine rule and area = ½ab sin C in any triangle. EXT
- Solve 3D problems by finding right-angled triangles inside solids. EXT
- Know the exact values of sin, cos and tan for 0°, 30°, 45°, 60° and 90°. EXT
- Sketch the graphs of sin x, cos x and tan x, and solve trigonometric equations. EXT

# 1 Pythagoras' theorem

## The theorem




**Pythagoras**

$c^2 = a^2 + b^2$
$25 = 9 + 16$
$5^2 = 3^2 + 4^2$
the square on the hypotenuse equals the sum of the squares on the other two sides

Build a square on each side. The two small squares (9 + 16) have exactly the same area as the big one (25).

**Hypotenuse** — the **longest side** of a right-angled triangle. It is always **opposite the right angle**.

**Pythagoras' theorem** — in a right-angled triangle, $c^2 = a^2 + b^2$, where **c** is the **hypotenuse**. It works **only** for right-angled triangles.

- **Pythagorean triples** are whole numbers that fit: **3, 4, 5** and **5, 12, 13** and **8, 15, 17** and **7, 24, 25**. Multiples work too, so **6, 8, 10** and **9, 12, 15** are right-angled.
- **Test for a right angle:** if the longest side squared equals the other two squared and added, the triangle **is** right-angled. This is called the **converse**.

WORKED EXAMPLE

**Do the sides 8 cm, 15 cm and 17 cm make a right-angled triangle?**

1. longest side = 17, so test $17^2 = 289$
2. $8^2 + 15^2 = 64 + 225 = 289$
3. **289 = 289, so yes, it is right-angled**

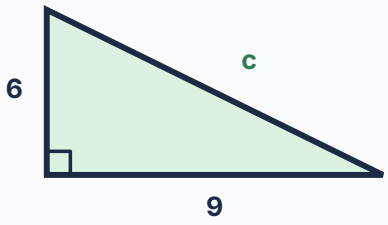
WORKED EXAMPLE

**Do the sides 6 cm, 7 cm and 10 cm make a right-angled triangle?**

1. $6^2 + 7^2 = 36 + 49 = 85$
2. $10^2 = 100$
3. **85 ≠ 100, so no**

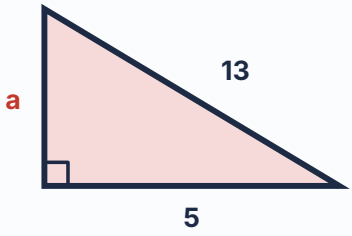
## Finding a missing side

### Longest side: ADD




$c^2 = 6^2 + 9^2$
$= 36 + 81 = 117$
$c = \sqrt{117} = 10.8$

### Shorter side: SUBTRACT




$a^2 = 13^2 - 5^2$
$= 169 - 25 = 144$
$a = \sqrt{144} = 12$

Always check: the hypotenuse is the longest side.

Look for the hypotenuse first. If you are finding it, add the squares. If you are finding a shorter side, subtract.

- **Step 1:** sketch the triangle and mark the **right angle**. The side opposite it is the hypotenuse **c**.
- **Step 2:** write $c^2 = a^2 + b^2$ and put the numbers in.
- **Step 3:** if you want a **shorter** side, rearrange to $a^2 = c^2 - b^2$.
- **Step 4:** take the **square root** at the end. Keep full calculator accuracy until the final answer.

WORKED EXAMPLE

**Find the hypotenuse of a right-angled triangle with shorter sides 6 cm and 9 cm.**

1. $c^2 = 6^2 + 9^2$
2. $= 36 + 81 = 117$
3. **$c = \sqrt{117} = 10.8$ cm (3 s.f.)**

WORKED EXAMPLE

**A right-angled triangle has hypotenuse 13 cm and one shorter side 5 cm. Find the other side.**

1. $a^2 = 13^2 - 5^2$
2. $= 169 - 25 = 144$
3. **$a = \sqrt{144} = 12$ cm**

WORKED EXAMPLE

**Find the hypotenuse when the shorter sides are 7.5 cm and 4 cm.**

1. $c^2 = 7.5^2 + 4^2 = 56.25 + 16 = 72.25$
2. **$c = \sqrt{72.25} = 8.5$ cm**

COMMON MISTAKE

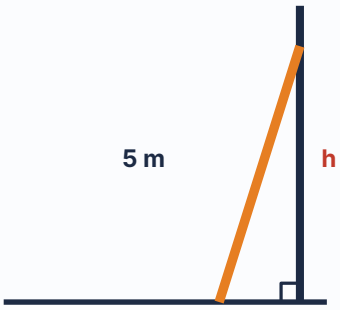
**Adding** when you should **subtract**, or the other way round. Quick check: the hypotenuse must be the **longest** side. If your answer for a shorter side is bigger than the hypotenuse, you added.

COMMON MISTAKE

Forgetting the **final square root** and giving $c^2$ as the answer.

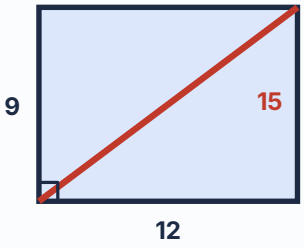
## Pythagoras in real problems

### Ladder



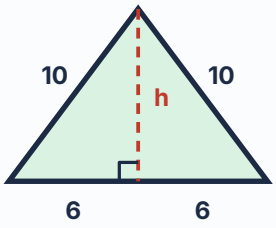

$h^2 = 5^2 - 1.5^2$
$h = 4.77$ m

### Rectangle




diagonal
$= \sqrt{(12^2 + 9^2)} = 15$

### Isosceles triangle




$h^2 = 10^2 - 6^2$, $h = 8$
area $= \frac{1}{2} \times 12 \times 8 = 48$

Most word problems hide a right-angled triangle: a ladder against a wall, the diagonal of a rectangle, or half of an isosceles triangle.

- **Ladders and walls:** the wall, the ground and the ladder make a right-angled triangle. The **ladder is the hypotenuse**.
- **Diagonal of a rectangle:** it splits the rectangle into two right-angled triangles. For a 12 cm by 9 cm rectangle the diagonal is $\sqrt{(12^2 + 9^2)} = 15$ cm.
- **Isosceles triangle:** drop the **perpendicular height** from the top. It splits the base in half and makes two right-angled triangles.
- **Distance between two points:** the horizontal and vertical differences are the two shorter sides, so $\mathbf{d = \sqrt{((x_2 - x_1)^2 + (y_2 - y_1)^2)}}$. This is the length formula from coordinate geometry.

WORKED EXAMPLE

**A 5 m ladder leans against a wall with its foot 1.5 m from the wall. How high up the wall does it reach?**

1. the ladder is the hypotenuse, so $h^2 = 5^2 - 1.5^2$
2. $= 25 - 2.25 = 22.75$
3. **$h = \sqrt{22.75} = 4.77$ m (3 s.f.)**

WORKED EXAMPLE

**An isosceles triangle has equal sides of 10 cm and a base of 12 cm. Find its height and its area.**

1. half the base = 6 cm
2. $h^2 = 10^2 - 6^2 = 64$, so $h = 8$ cm
3. **area $= \frac{1}{2} \times 12 \times 8 = 48$ cm²**

WORKED EXAMPLE

**Find the distance between the points (1, 2) and (7, 10).**

1. horizontal difference $= 7 - 1 = 6$
2. vertical difference $= 10 - 2 = 8$
3. **$d = \sqrt{(6^2 + 8^2)} = \sqrt{100} = 10$**

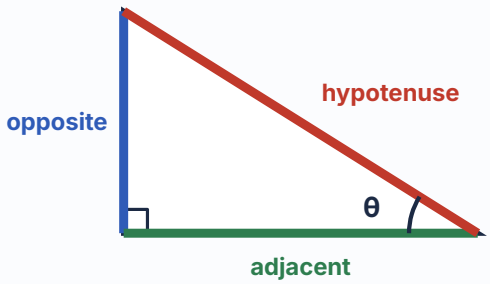
EXAM TIP

In **"show that"** questions give the **exact** form first (for example $\sqrt{117}$) and then the rounded value. Never round in the middle of a calculation.

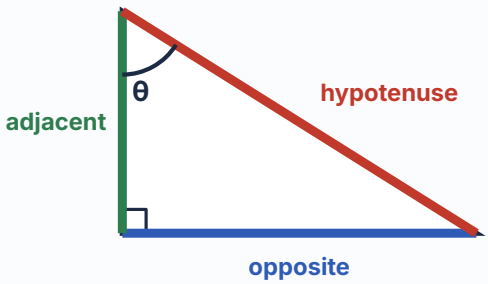
## 2 Trigonometry in right-angled triangles

### Naming the sides

**θ at the bottom right**




**θ at the top**




The opposite is across from θ. The adjacent is next to θ.
The hypotenuse is opposite the right angle (the longest side).

The hypotenuse never changes. The opposite and adjacent sides swap over when you move to the other angle.

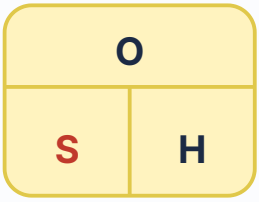
**Opposite** — the side **across** from the angle θ. It does not touch θ.

**Adjacent** — the side **next to** θ that is **not** the hypotenuse.

- **Label in this order:** hypotenuse first (opposite the right angle), then the opposite (across from θ), and the adjacent is what is left.
- **The angle decides the labels.** The same side can be "opposite" for one angle and "adjacent" for the other.

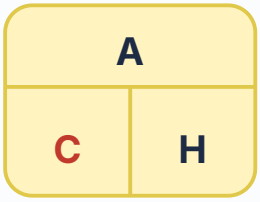
### SOH CAH TOA

**SOH**



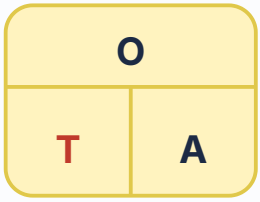

$\sin\theta = O \div H$
$O = H \times \sin\theta$
$H = O \div \sin\theta$

**CAH**




$\cos\theta = A \div H$
$A = H \times \cos\theta$
$H = A \div \cos\theta$

**TOA**




$\tan\theta = O \div A$
$O = A \times \tan\theta$
$A = O \div \tan\theta$

Cover the one you want: side by side means × and one above the other means ÷.
O = opposite, A = adjacent, H = hypotenuse, S = sin, C = cos, T = tan

Cover up the quantity you want. If the other two are side by side, multiply. If one is above the other, divide.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

SOH

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

CAH

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

TOA

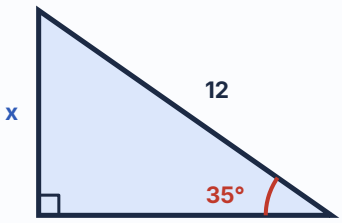
- **Pick the ratio** by looking at the two sides involved: the one you **have** and the one you **want**. If they are O and H use sin, A and H use cos, O and A use tan.
- **The ratios depend only on the angle**, not on the size of the triangle. A bigger triangle with the same angles has the same sin, cos and tan.
- **Sine and cosine** of an angle are never more than 1, because the hypotenuse is the longest side.

**MEMORY TRICK**

**S**ome **O**ld **H**ens **C**an't **A**lways **H**ide **T**heir **O**ld **A**ge. The first letters spell SOH CAH TOA.

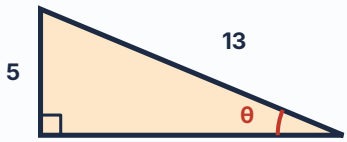
## Finding sides and angles

### Find a side




**sin 35° = x ÷ 12**
**x = 12 × sin 35°**
**x = 6.88**

### Find an angle




**sin θ = 5 ÷ 13**
**θ = sin⁻¹(5 ÷ 13)**
**θ = 22.6°**

Calculator in DEG mode. Use $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ to find an angle.

To find a side, multiply (or divide) by the ratio. To find an angle, use the inverse button on your calculator.

- **Finding a side:** write the ratio with numbers, then rearrange. If the unknown is on **top** of the fraction, **multiply**. If it is on the **bottom**, **divide**.
- **Finding an angle:** work out the ratio as a number, then use **$\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$**. For example, if $\sin\theta = 5/13$, then $\theta = \sin^{-1}(5 \div 13)$.
- **Check the mode.** Your calculator must be in **degrees** (DEG). Radians give wrong answers.

WORKED EXAMPLE

**Find x when the hypotenuse is 12 cm and the angle opposite x is 35°.**

1. x is opposite and 12 is the hypotenuse, so use sin
2. sin 35° = x ÷ 12
3. **x = 12 × sin 35° = 6.88 cm**

WORKED EXAMPLE

**Find x when the opposite side is 9 cm and the angle is 52°, with x being the adjacent side.**

1. O and A, so use tan
2. tan 52° = 9 ÷ x
3. **x = 9 ÷ tan 52° = 7.03 cm (x is on the bottom, so divide)**

WORKED EXAMPLE

**Find the hypotenuse h when the adjacent side is 8 cm and the angle is 40°.**

1. A and H, so use cos
2. cos 40° = 8 ÷ h
3. **h = 8 ÷ cos 40° = 10.4 cm**

WORKED EXAMPLE

**Find angle θ when the opposite side is 5 cm and the hypotenuse is 13 cm.**

1. sin θ = 5 ÷ 13
2. **θ = $\sin^{-1}(5 \div 13) = 22.6°$**

WORKED EXAMPLE

**Find angle θ when the opposite side is 7 cm and the adjacent side is 4 cm.**

1. tan θ = 7 ÷ 4 = 1.75
2. **θ = $\tan^{-1}(1.75) = 60.3°$**

WORKED EXAMPLE

**An isosceles triangle has equal sides of 10 cm and base angles of 70°. Find the base and the area.**

1. drop the height to split the base in two. Half the base = 10 × cos 70° = 3.420
2. base = 2 × 3.420 = **6.84 cm**
3. height = 10 × sin 70° = 9.397
4. **area = ½ × 6.840 × 9.397 = 32.1 $cm^2$**

COMMON MISTAKE

Calculator in **radian** mode. Check that it shows **D** or **DEG** at the top.

COMMON MISTAKE

**Rounding in the middle.** Round only at the end, to **3 significant figures** (or 1 decimal place for angles).

COMMON MISTAKE

Mixing up **opposite** and **adjacent**. Always label first, from the angle you are using.

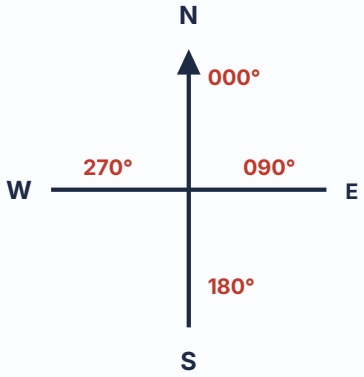
EXAM TIP

**Pythagoras or trig?** If you have **two sides and want the third**, use Pythagoras. If an **angle** is involved (given or wanted), use **SOH CAH TOA**.

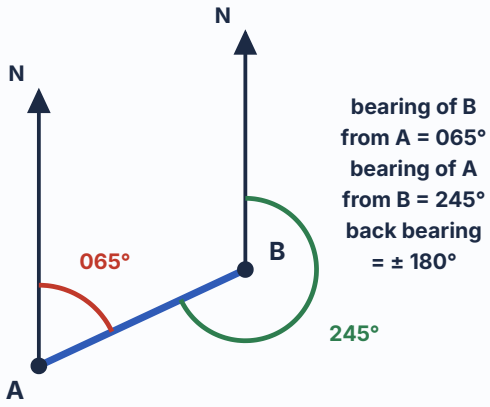
Riyaz Sir

## 3 Bearings, elevation and depression

### Three-figure bearings




Three figures, measured clockwise from North




Start at North, turn clockwise to the direction of travel, and write the angle with three figures.

**Bearing** — the direction of one point from another, measured **clockwise from North** and written with **three figures**. East is 090°, South is 180° and West is 270°.

**Back bearing** — the bearing for the **return journey**. Add or subtract **180°**.

- **Draw a North line at the point you are going FROM.** The word "of B from A" means start at A.
- **Three figures always:** write **065°**, never 65°.
- **Back bearing:** if the bearing is **less than 180°**, add 180°. If it is **more than 180°**, subtract 180°.
- **Using trigonometry:** draw the triangle with the North line as one side. The angle from North is usually the angle in the triangle.

WORKED EXAMPLE

**B is 6 km east and 8 km north of A. Find the distance AB and the bearing of B from A.**

1. AB = $\sqrt{(6^2 + 8^2)}$ = **10 km**
2. angle from North: tan θ = 6 ÷ 8 = 0.75
3. θ = $\tan^{-1}(0.75)$ = 36.87°
4. **bearing = 037°**

WORKED EXAMPLE

**The bearing of B from A is 065°. Find the bearing of A from B.**

1. **065 + 180 = 245°**

WORKED EXAMPLE

**A ship sails 15 km on a bearing of 120°. How far east and how far south of its start is it?**

1. 120° is 60° past East (090°), so the angle from South is 60°
2. east = 15 × sin 60° = **13.0 km**
3. **south = 15 × cos 60° = 7.5 km**

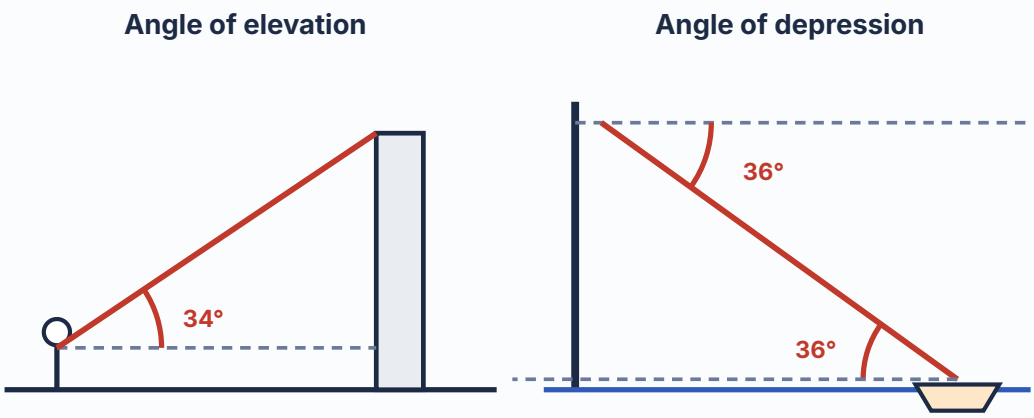
COMMON MISTAKE

Measuring from the **wrong North line**. The bearing of A **from** B uses the North line at **B**.

COMMON MISTAKE

Measuring **anticlockwise**, or giving only two figures. Bearings are always clockwise and three digits.

## Angles of elevation and depression




**looking UP from the horizontal**
the line of sight is above it

**looking DOWN from the horizontal**
angle of depression = angle of elevation (alternate angles)

Both angles are measured from the horizontal. Elevation looks up and depression looks down. They are equal because they are alternate angles.

**Angle of elevation** — the angle **up** from the horizontal to a line of sight.

**Angle of depression** — the angle **down** from the horizontal to a line of sight.

- **Always draw a right-angled triangle** with a horizontal line and a vertical line.
- **The angle of depression is not inside the triangle at the top.** The matching angle **inside the triangle** is at the bottom and is equal to it (alternate angles).

WORKED EXAMPLE

**From a point 40 m from the foot of a tower, the angle of elevation of the top is 32°. Find the height of the tower.**

1. O and A, so tan 32° = h ÷ 40
2. **h = 40 × tan 32° = 25.0 m**

WORKED EXAMPLE

**From the top of a 50 m cliff the angle of depression of a boat is 18°. How far is the boat from the foot of the cliff?**

1. the angle inside the triangle at the boat is also 18°
2. tan 18° = 50 ÷ d
3. **d = 50 ÷ tan 18° = 154 m**

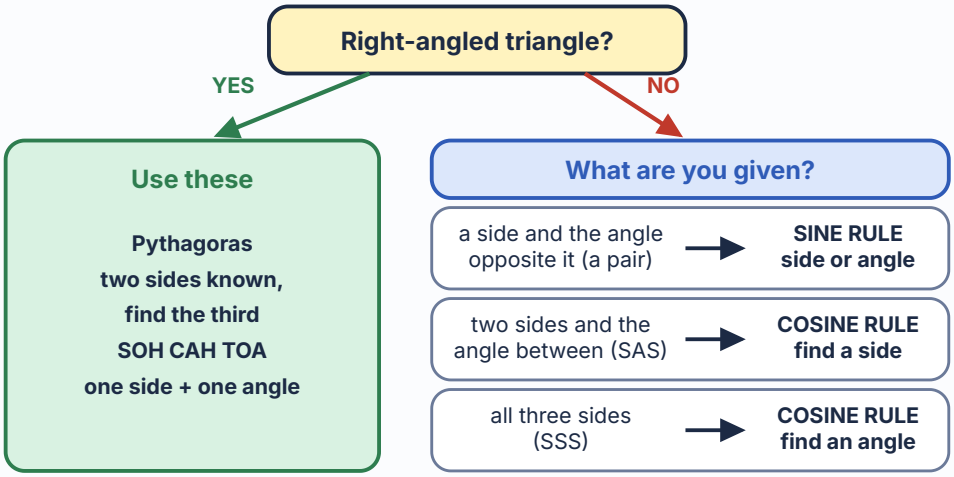
COMMON MISTAKE

Measuring the angle of depression from the **vertical**. It is always from the **horizontal**.

## 4 Sine and cosine rules EXT

### Which rule?

EXTENDED



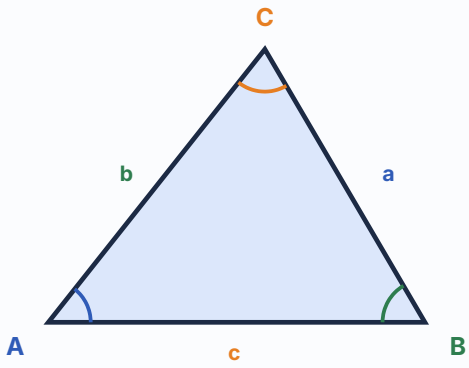

**Always sketch and label the triangle first.**

Ask whether the triangle has a right angle. If not, look at what you are given: a pair, two sides and the angle between, or three sides.

- **Labelling:** the angles are **A, B, C** and the sides opposite them are **a, b, c**. So side **a** is opposite angle **A**.
- **A pair** means a side and its **opposite** angle, such as a and A. If you have a pair, use the **sine rule**.
- **SAS** means two sides and the angle **between** them. Use the **cosine rule** to find the third side.
- **SSS** means all three sides and no angle. Use the **cosine rule** to find an angle.
- **Two angles given?** Find the third using **angles add up to 180°**, then you have a pair.

### The sine rule

EXTENDED




Each side is matched with the angle opposite it (same colour)

**The sine rule**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use it when you know a PAIR:
a side and the angle opposite it.
Only two of the fractions
are used in any one question.

**Side? Unknown on top:**
**$x \div \sin 40° = 8 \div \sin 65° \rightarrow x = 5.67$**

**Angle? sin on top:**
**$\sin \theta \div 6 = \sin 50° \div 9 \rightarrow \theta = 30.7°$**

Each side is divided by the sine of the angle opposite it. All three fractions are equal.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

use to find a side

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

use to find an angle

- **Use only two** of the three fractions in any question. Choose the two that contain the pair you know and the thing you want.
- **To find a side:** put the **unknown on top**. **To find an angle:** put **sin on top**.
- **Obtuse angles:** $\sin^{-1}$ gives an **acute** angle. If the diagram shows an **obtuse** angle, the answer is **180° minus** your calculator value.

**WORKED EXAMPLE**

**Find x if x ÷ sin 40° = 8 ÷ sin 65°.**

1. x = 8 × sin 40° ÷ sin 65°
2. **= 8 × 0.6428 ÷ 0.9063 = 5.67 cm**

**WORKED EXAMPLE**

**In triangle ABC, angle A = 48°, angle B = 71° and a = 12 cm. Find b.**

1. pair is a and A, and b is opposite B
2. b ÷ sin 71° = 12 ÷ sin 48°
3. **b = 12 × sin 71° ÷ sin 48° = 15.3 cm**

**WORKED EXAMPLE**

**In the same triangle find c.**

1. angle C = 180° − 48° − 71° = 61°
2. **c = 12 × sin 61° ÷ sin 48° = 14.1 cm**

**WORKED EXAMPLE**

**Find angle θ if sin θ ÷ 6 = sin 50° ÷ 9, and θ is acute.**

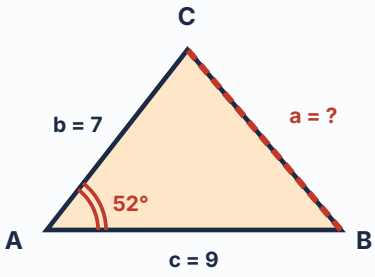
1. sin θ = 6 × sin 50° ÷ 9 = 0.5107
2. **θ = $\sin^{-1}$(0.5107) = 30.7°**

**COMMON MISTAKE**

Using a side and an angle that are **not opposite** each other. The side and angle in a pair must be across from each other.

## The cosine rule

EXTENDED




**The cosine rule**

$$a^2 = b^2 + c^2 - 2bc\cos A$$

find a SIDE (two sides and the angle between)

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

find an ANGLE (all three sides)

The side on the left (a) and the angle inside the cos (A) are always opposite each other.

**Here: $a^2 = 7^2 + 9^2 - 2 \times 7 \times 9 \times \cos 52° = 52.4$**
**so a = 7.24**

Use it when the angle is between the two known sides (SAS), or when you know all three sides (SSS).

$$a^2 = b^2 + c^2 - 2bc\cos A$$

to find a side

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

to find an angle

- **The side on the left (a) and the angle in the cos (A) are opposite each other.** The other two sides, b and c, are the ones around angle A.
- **Order of working:** do **2 × b × c × cos A** first, then subtract from $b^2 + c^2$. Do **not** subtract before multiplying by cos A.
- **A negative cosine** means the angle is **obtuse** (bigger than 90°). The calculator gives it correctly.

WORKED EXAMPLE

**Find a when b = 7 cm, c = 9 cm and A = 52°.**

1. $a^2 = 7^2 + 9^2 - 2 \times 7 \times 9 \times \cos 52°$
2. $= 49 + 81 - 126 \times 0.6157$
3. $= 130 - 77.57 = 52.43$
4. **$a = 7.24$ cm**

WORKED EXAMPLE

**Triangle with sides 5 cm, 7 cm and 10 cm. Find the largest angle.**

1. largest angle is opposite 10, so a = 10, b = 5, c = 7
2. $\cos A = (5^2 + 7^2 - 10^2) \div (2 \times 5 \times 7) = -26 \div 70 = -0.3714$
3. **$A = \cos^{-1}(-0.3714) = 111.8°$ (obtuse, as the cosine is negative)**

WORKED EXAMPLE

**Triangle with sides 6 cm, 7 cm and 8 cm. Find the largest angle.**

1. the angle is opposite 8
2. $\cos A = (6^2 + 7^2 - 8^2) \div (2 \times 6 \times 7) = 21 \div 84 = 0.25$
3. **$A = \cos^{-1}(0.25) = 75.5°$**

COMMON MISTAKE

Writing **$(b^2 + c^2 - 2bc)\cos A$**. The cos only multiplies the **2bc** part.

EXAM TIP

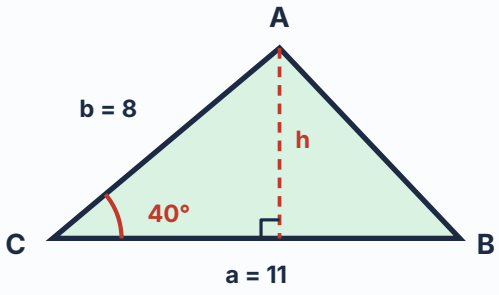
**Largest angle first** is a good check. It is opposite the **longest side**. If it is obtuse the cosine is negative.

## Area of a triangle with sine

EXTENDED

**Area = ½ ab sin C**

base a, height h = b sin C,
so area = ½ × a × b sin C
The angle C must be BETWEEN
the two sides a and b.




sin C turns a slanted side into a height

**a = 11, b = 8, C = 40°**
**Area = ½ × 11 × 8 × sin 40°**
**= 44 × 0.6428**
**= 28.3 cm²**

The height is b sin C, so the area is half the base times that height. The angle must be between the two sides.

$$\text{Area} = \tfrac{1}{2}\,ab\sin C$$

a and b are two sides, C is the angle **between** them

- **Use this when you do not have the perpendicular height.** The sine rule gives the height for you.
- **Works for obtuse angles too.** For C = 150°, sin 150° = 0.5.
- **Finding an angle:** rearrange to sin C = 2 × area ÷ (ab). Use the diagram to decide if C is acute or obtuse.

**WORKED EXAMPLE**

**Find the area of a triangle with sides 11 cm and 8 cm and an angle of 40° between them.**

1. area = ½ × 11 × 8 × sin 40°
2. **= 44 × 0.6428 = 28.3 cm²**

**WORKED EXAMPLE**

**Triangle ABC has AB = 9 cm, AC = 11 cm and angle A = 64°. Find its area and the length BC.**

1. area = ½ × 9 × 11 × sin 64° = 49.5 × 0.8988 = **44.5 cm²**
2. $BC^2 = 9^2 + 11^2 - 2 \times 9 \times 11 \times \cos 64° = 202 - 198 \times 0.4384 = 115.2$
3. **BC = 10.7 cm**

**WORKED EXAMPLE**

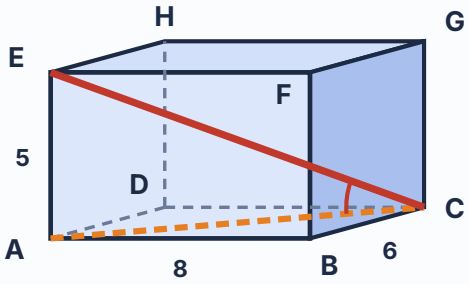
**A triangle has area 30 cm², sides 10 cm and 8 cm and an acute angle C between them. Find C.**

1. ½ × 10 × 8 × sin C = 30
2. 40 sin C = 30, so sin C = 0.75
3. **$C = \sin^{-1}(0.75) = 48.6°$**

## 5 Trigonometry in 3D EXT

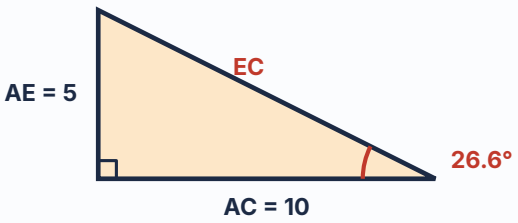
### Cuboids and the space diagonal

EXTENDED




space diagonal EC (red)

**Step 1: base diagonal AC**
**$AC^2 = 8^2 + 6^2 = 100$, AC = 10**
**Step 2: space diagonal EC**
**$EC^2 = 10^2 + 5^2 = 125$, EC = 11.2**
**Step 3: angle between EC and the base**
**$\tan\theta = 5 \div 10$, so θ = 26.6°**




Find a right-angled triangle inside the solid. Base diagonal first, then the diagonal through the solid.

**Angle between a line and a plane** — the angle between the line and its **shadow** (projection) on the plane. For a diagonal of a cuboid, the shadow on the base is the **base diagonal**.

- **Method:** (1) pick the right-angled triangle that contains your line, (2) **redraw it flat** with its true lengths, (3) use Pythagoras or SOH CAH TOA.
- **Base diagonal:** √(length² + width²). **Space diagonal:** √(length² + width² + height²).
- **The angle between the diagonal and the base** is found using the **height** (opposite) and the **base diagonal** (adjacent).

WORKED EXAMPLE

**A cuboid is 8 cm by 6 cm by 5 cm. Find the space diagonal and the angle it makes with the base.**

1. base diagonal = $\sqrt{(8^2 + 6^2)} = \sqrt{100} = 10$ cm
2. space diagonal = $\sqrt{(10^2 + 5^2)} = \sqrt{125} =$ **11.2 cm**
3. $\tan\theta = 5 \div 10$
4. **$\theta = 26.6°$**

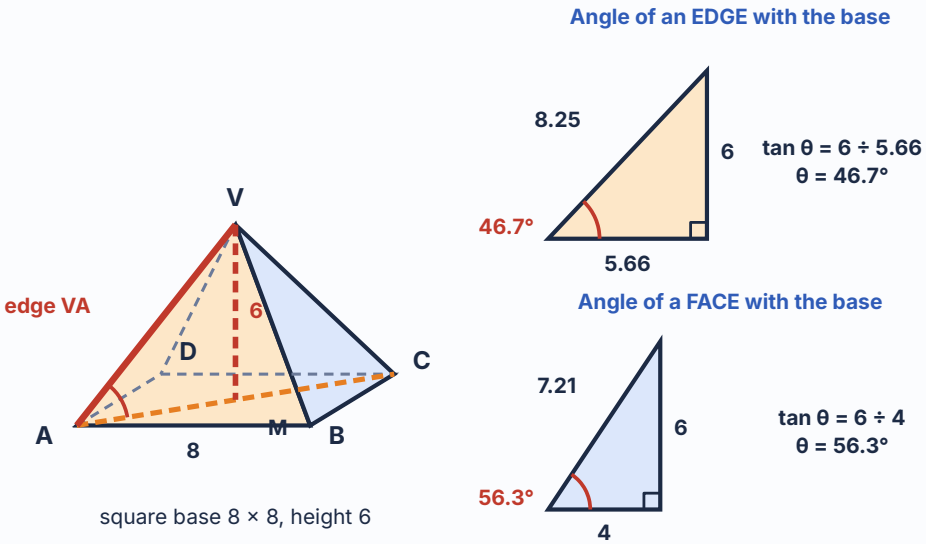
WORKED EXAMPLE

**A cuboid has AB = 9 cm, BC = 12 cm and CG = 8 cm. Find AG.**

1. AC = $\sqrt{(9^2 + 12^2)} = 15$ cm
2. **AG = $\sqrt{(15^2 + 8^2)} = \sqrt{289} = 17$ cm**

## Pyramids

EXTENDED




The apex sits directly above the centre of the base. Two different right-angled triangles give the angle of an edge and the angle of a face.

- **Angle of an edge with the base:** use the triangle with the **height**, the **half-diagonal** of the base and the **edge**.
- **Angle of a face with the base:** use the triangle with the **height**, **half a side** of the base and the **slant height** of the face.
- **For a square base of side s:** the half-diagonal is $s\sqrt{2} \div 2$, and half a side is $s \div 2$.

WORKED EXAMPLE

**A square-based pyramid has base 8 cm by 8 cm and height 6 cm. Find the length of a sloping edge and the angle it makes with the base.**

1. half-diagonal = $\frac{1}{2} \times 8\sqrt{2} = 4\sqrt{2} = 5.657$ cm
2. edge$^2$ = $5.657^2 + 6^2 = 32 + 36 = 68$
3. edge = **8.25 cm**
4. **$\tan\theta = 6 \div 5.657$, so $\theta = 46.7°$**

WORKED EXAMPLE

**For the same pyramid find the angle between a triangular face and the base.**

1. half a side = 4 cm, height = 6 cm
2. tan θ = 6 ÷ 4 = 1.5
3. **θ = 56.3°**

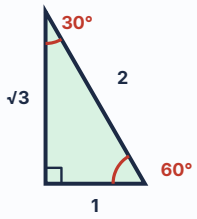
COMMON MISTAKE

Using the **edge** where you need the **half-diagonal**, or the **side** where you need the **half-diagonal** of the base. Draw the flat triangle every time.

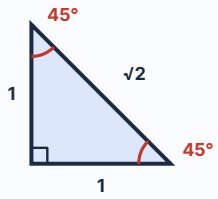
## 6 Trigonometric graphs and equations EXT

### Exact values

EXTENDED




half an equilateral triangle of side 2




a square cut along its diagonal

**Learn the two triangles and you can rebuild the whole table**

| θ | 0° | 30° | 45° | 60° | 90° |
|---|---|---|---|---|---|
| sin θ | 0 | 1/2 | √2/2 | √3/2 | 1 |
| cos θ | 1 | √3/2 | √2/2 | 1/2 | 0 |
| tan θ | 0 | √3/3 | 1 | √3 | — |

tan 90° is undefined. √3/3 is the same as 1/√3.

Two special triangles give every exact value you need. Half an equilateral triangle gives 30° and 60°. A square cut along its diagonal gives 45°.

- **Pattern for sin:** the values for 0°, 30°, 45°, 60°, 90° are √0 ÷ 2, √1 ÷ 2, √2 ÷ 2, √3 ÷ 2 and √4 ÷ 2. The cosines are the same list in **reverse**.
- **For tan:** divide sin by cos. So tan 30° = (1/2) ÷ (√3/2) = 1/√3 and tan 60° = √3.
- **Exact means no decimals.** Leave surds and fractions in your answer when the question says "exact".

WORKED EXAMPLE

**Find the exact value of sin 60° × cos 30°.**

1. sin 60° = √3/2 and cos 30° = √3/2
2. **product = (√3 × √3) ÷ 4 = 3/4**

WORKED EXAMPLE

**An equilateral triangle has sides 6 cm. Find its exact height.**

1. the height splits it into a 30°-60°-90° triangle with hypotenuse 6
2. **h = 6 × sin 60° = 6 × √3/2 = 3√3 cm**

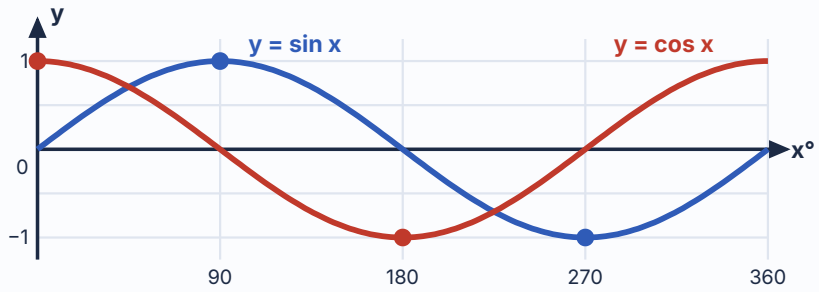
WORKED EXAMPLE

**Find the exact length x if the hypotenuse is 10 cm and the angle opposite x is 30°.**

1. **x = 10 × sin 30° = 10 × 1/2 = 5 cm**

## Graphs of sin, cos and tan

EXTENDED




**sin x**
**starts at 0, peak 1 at 90°**
**zero at 0°, 180° and 360°**
**minimum −1 at 270°**

**cos x**
**starts at 1, zero at 90°**
**minimum −1 at 180°**
**zero again at 270°, back to 1 at 360°**

**Both repeat every 360° and stay between −1 and 1.**

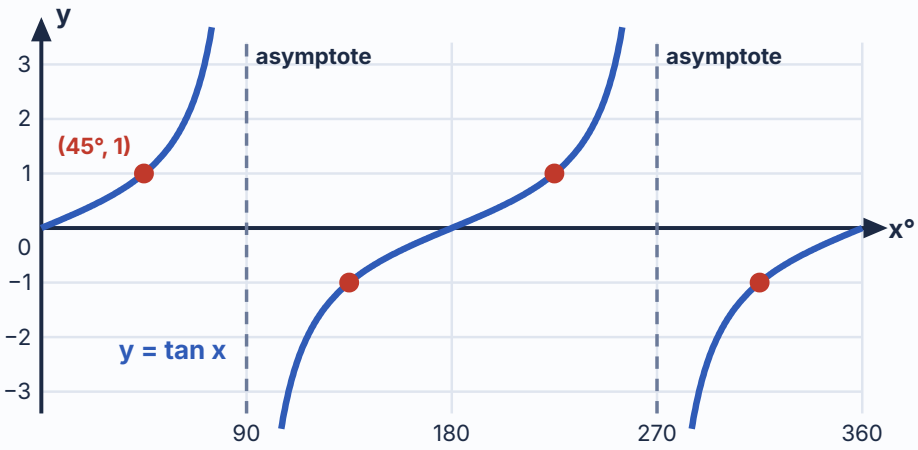
Both waves repeat every 360° and stay between −1 and 1. The cosine curve is the sine curve shifted 90° to the left.

- **Key points of y = sin x:** (0°, 0), (90°, 1), (180°, 0), (270°, −1), (360°, 0).
- **Key points of y = cos x:** (0°, 1), (90°, 0), (180°, −1), (270°, 0), (360°, 1).
- **Period** is how far you go before the shape repeats. For sin and cos it is **360°**. **Amplitude** is the maximum height above the middle, which is **1** for both.
- **The curves cross** where sin x = cos x, at **45°** and **225°**.

Riyaz Sir

## The tangent graph

EXTENDED




tan x repeats every 180° and passes through 0° and 180°.
It has asymptotes (lines it never touches) at 90° and 270°.
Unlike sin and cos, it is not limited to between −1 and 1.

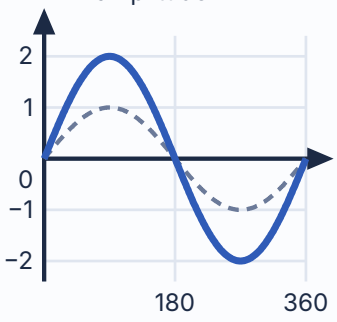
The tangent curve has vertical asymptotes at 90° and 270°. It repeats every 180° and has no maximum or minimum.

- **Asymptotes** are lines the curve gets closer and closer to, but never touches. For y = tan x they are x = 90° and x = 270°.
- **Key points:** tan 0° = 0, tan 45° = 1, tan 135° = −1, tan 180° = 0.
- **Between** the asymptotes the curve rises from −∞ to +∞, so every value of tan x happens exactly once in each cycle.

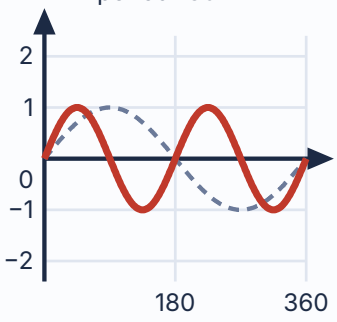
## Transformed trig graphs

EXTENDED

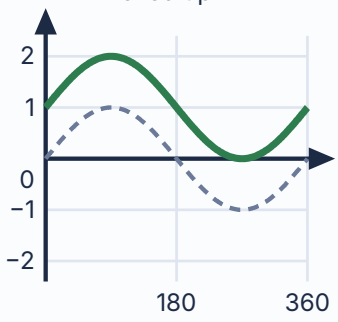
**y = 2 sin x**
amplitude 2




**y = sin 2x**
period 180°




**y = sin x + 1**
moved up 1




**y = a sin bx + c**

| a = amplitude (the height) | b = cycles in 360° period = 360° ÷ b | c = slide up or down |
|---|---|---|

In y = a sin bx + c, the number a stretches the height, b squashes the width, and c slides the whole graph up or down.

- **a is the amplitude.** y = 2 sin x goes up to 2 and down to −2.
- **b changes the period.** The period is **360° ÷ b**. So y = sin 2x repeats every 180°.
- **c moves the graph up (c positive) or down.** y = sin x + 1 goes from 0 to 2.

WORKED EXAMPLE

**State the amplitude and the period of $y = 3 \cos 2x$.**

1. amplitude = **3**
2. **period = 360° ÷ 2 = 180°**

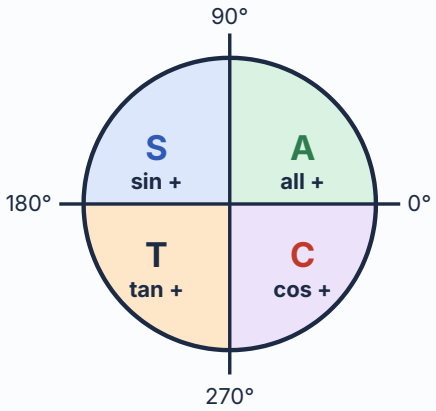
WORKED EXAMPLE

**Find the maximum and minimum values of $y = 2 \sin x - 1$.**

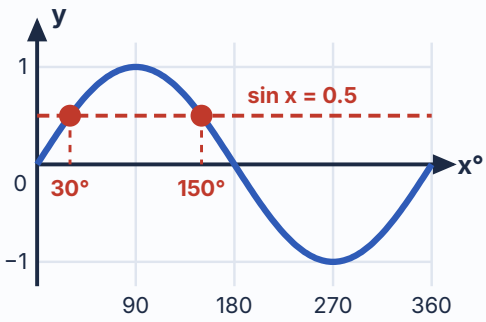
1. sin x lies between −1 and 1
2. 2 sin x lies between −2 and 2
3. **subtract 1: maximum = 1 and minimum = −3**

## Solving trigonometric equations

EXTENDED




**sin: x and 180° − x**
**cos: x and 360° − x**
**tan: x and 180° + x**




**Solve sin x = 0.5 for 0° ≤ x ≤ 360°**
**1. calculator: $\sin^{-1} 0.5 = 30°$**
**2. sin is + in quadrants 1 and 2**
**3. second answer: 180° − 30° = 150°**

**x = 30° or x = 150°**

Use the calculator for the first answer, then use the CAST diagram or the graph to find the second answer in the range 0° to 360°.

- **Step 1:** get the trig function **alone**: for example 2 sin x = 1.4 becomes sin x = 0.7.
- **Step 2:** use the **inverse** on the calculator to find the first answer.
- **Step 3:** find the **second answer** using the graph or CAST. **Sin:** x and 180° − x. **Cos:** x and 360° − x. **Tan:** x and 180° + x.
- **Negative answer from the calculator?** Add 360° (or 180° for tan) to make it fit the range.
- **How many solutions?** For sin x = k or cos x = k with k between −1 and 1 there are **two** in 0° to 360°. If k = 1 or −1 there is **one**. If k is bigger than 1 there are **none**.

WORKED EXAMPLE

**Solve $\sin x = 0.5$ for $0° \le x \le 360°$.**

1. $\sin^{-1}(0.5) = 30°$
2. second answer: 180° − 30° = 150°
3. **x = 30° or 150°**

WORKED EXAMPLE

**Solve $\cos x = -0.5$ for $0° \le x \le 360°$.**

1. $\cos^{-1}(-0.5) = 120°$
2. second answer: 360° − 120° = 240°
3. **x = 120° or 240°**

WORKED EXAMPLE

**Solve tan x = −1 for 0° ≤ x ≤ 360°.**

1. $\tan^{-1}(-1) = -45°$
2. add 180° to get 135°
3. add 360° to get 315°
4. **x = 135° or 315°**

WORKED EXAMPLE

**Solve 2 sin x = 1.4 for 0° ≤ x ≤ 360°.**

1. sin x = 0.7
2. first answer = $\sin^{-1}(0.7) = 44.4°$
3. second = 180° − 44.4° = 135.6°
4. **x = 44.4° or 135.6°**

WORKED EXAMPLE

**Solve sin x = −0.5 for 0° ≤ x ≤ 360°.**

1. $\sin^{-1}(-0.5) = -30°$
2. sin is negative in quadrants 3 and 4
3. x = 180° + 30° = 210° and x = 360° − 30° = 330°
4. **x = 210° or 330°**

COMMON MISTAKE

Giving **only one answer**. Always check for a second one in the range.

EXAM TIP

**Check on the graph.** Draw a horizontal line at the value on the sine or cosine curve. It cuts the curve at your solutions.

Riyaz Sir

## 7 Test yourself

### Test yourself — cover the answers first

**QUICK CHECK**

1. Find the hypotenuse of a right-angled triangle with shorter sides 9 cm and 12 cm.
2. A right-angled triangle has hypotenuse 17 cm and one shorter side 8 cm. Find the other side.
3. Is a triangle with sides 7 cm, 24 cm and 25 cm right-angled?
4. Find the length of the diagonal of a rectangle 5 cm by 12 cm.
5. Find the distance between the points (2, 1) and (8, 9).
6. A 6 m ladder rests against a wall with its foot 2 m from the wall. How high up the wall does it reach? Give your answer to 3 significant figures.
7. Name the longest side of a right-angled triangle.
8. Find angle θ if $\sin\theta = 0.5$.
9. Find x when the hypotenuse is 20 cm and the angle opposite x is 30°.
10. Find the adjacent side when the hypotenuse is 15 cm and the angle is 60°.
11. Find angle θ when the opposite and adjacent sides are both 8 cm.
12. Find angle θ when the opposite side is 5 cm and the hypotenuse is 9 cm.
13. A tree casts a shadow 12 m long when the angle of elevation of the Sun is 40°. Find the height of the tree.
14. Write down the three-figure bearing for due West.
15. The bearing of B from A is 050°. Find the bearing of A from B.
16. The bearing of Q from P is 285°. Find the bearing of P from Q.
17. The angle of elevation of A from B is 25°. What is the angle of depression of B from A?
18. EXT Write down the exact value of cos 60°.
19. EXT Write down the exact value of tan 45°.
20. EXT In triangle ABC, a = 10 cm, A = 30° and B = 45°. Find b.
21. EXT Write down the cosine rule for finding side a.
22. EXT Find the area of a triangle with sides 6 cm and 9 cm and an angle of 30° between them.
23. EXT State the amplitude and the period of $y = 2\sin 3x$.
24. EXT Solve $\cos x = 0.5$ for $0° \le x \le 360°$.
25. EXT Find the length of the space diagonal of a cuboid 3 cm by 4 cm by 12 cm.

**EXAM-STYLE**

26. The foot of a ladder is 1.8 m from a wall and its top reaches 6 m up the wall. Find the length of the ladder and the angle it makes with the ground. [4]
27. An isosceles triangle has equal sides of 13 cm and a base of 10 cm. Find its height, its area and a base angle. [5]
28. From the top of a 45 m building the angle of depression of a car is 28°. Find how far the car is from the foot of the building. [3]
29. A ship sails 12 km on a bearing of 040° from P to Q, then 9 km on a bearing of 130° to R. Show that angle PQR = 90°, then find PR and the bearing of R from P. [6]
30. EXT In triangle ABC, AB = 8 cm, BC = 11 cm and angle ABC = 70°. Find AC and the area of the triangle. [5]
31. EXT In triangle PQR, PQ = 7 cm, QR = 9 cm and PR = 12 cm. Find the largest angle. [3]
32. EXT In triangle ABC, A = 35°, B = 80° and AB = 14 cm. Find BC. [4]
33. EXT A pyramid has a rectangular base 12 cm by 5 cm. The apex is 8 cm directly above the centre of the base. Find the length of a sloping edge and the angle it makes with the base. [5]

34. EXT A cuboid ABCDEFGH has AB = 9 cm, BC = 12 cm and CG = 8 cm. Find AG and the angle between AG and the base. [4]

35. EXT Solve $3 \sin x + 1 = 0$ for $0° \leq x \leq 360°$. [3]

36. EXT Sketch $y = 2 \sin x$ for $0° \leq x \leq 360°$, marking the maximum, minimum and where the curve crosses the x-axis. [3]

Riyaz Sir

**Show answers**

1. 15 cm
2. 15 cm
3. Yes: $49 + 576 = 625 = 25^2$
4. 13 cm
5. 10
6. 5.66 m
7. hypotenuse
8. 30°
9. 10 cm
10. 7.5 cm
11. 45°
12. 33.7°
13. 10.1 m
14. 270°
15. 230°
16. 105°
17. 25°
18. 1/2
19. 1
20. 14.1 cm
21. $a^2 = b^2 + c^2 - 2bc \cos A$
22. 13.5 $cm^2$
23. amplitude 2, period 120°
24. 60° and 300°
25. 13 cm
26. length = $\sqrt{(1.8^2 + 6^2)} = \sqrt{39.24} = 6.26$ m
    $\tan \theta = 6 \div 1.8$
    $\theta = \tan^{-1}(3.333) = 73.3°$
27. half the base = 5 cm
    $h = \sqrt{(13^2 - 5^2)} = \sqrt{144} = 12$ cm
    area = $\frac{1}{2} \times 10 \times 12 = 60$ $cm^2$
    $\cos \theta = 5 \div 13$, so $\theta = 67.4°$
28. angle inside the triangle at the car = 28°
    $\tan 28° = 45 \div d$
    $d = 45 \div \tan 28° = 84.6$ m
29. back bearing of PQ = 040° + 180° = 220°
    angle PQR = 220° − 130° = 90°
    $PR = \sqrt{(12^2 + 9^2)} = 15$ km
    angle QPR: tan = 9 ÷ 12, so 36.9°
    bearing of R from P = 040° + 36.9° = 077°

30. $AC^2 = 8^2 + 11^2 - 2 \times 8 \times 11 \times \cos 70° = 185 - 60.20 = 124.8$
    $AC = 11.2$ cm
    area $= \frac{1}{2} \times 8 \times 11 \times \sin 70° = 41.3$ cm$^2$

31. largest angle is opposite 12, so it is angle Q
    $\cos Q = (7^2 + 9^2 - 12^2) \div (2 \times 7 \times 9) = -14 \div 126 = -0.1111$
    $Q = 96.4°$

32. angle C $= 180° - 35° - 80° = 65°$
    pair: AB is opposite C, and BC is opposite A
    $BC \div \sin 35° = 14 \div \sin 65°$
    $BC = 14 \times \sin 35° \div \sin 65° = 8.86$ cm

33. base diagonal $= \sqrt{(12^2 + 5^2)} = 13$ cm, half $= 6.5$ cm
    edge $= \sqrt{(6.5^2 + 8^2)} = \sqrt{106.25} = 10.3$ cm
    $\tan \theta = 8 \div 6.5$, so $\theta = 50.9°$

34. AC $= 15$ cm
    AG $= \sqrt{(15^2 + 8^2)} = 17$ cm
    $\tan \theta = 8 \div 15$
    $\theta = 28.1°$

35. $\sin x = -1/3$
    $\sin^{-1}(-1/3) = -19.5°$
    $x = 180° + 19.5° = 199.5°$ and $x = 360° - 19.5° = 340.5°$

36. wave starting at the origin
    maximum (90°, 2) and minimum (270°, −2)
    crosses the x-axis at 0°, 180° and 360°

**Stuck on a question?**

Send it to me on WhatsApp. I read every message.

Want regular help? Send this to your parent →

Share with a friend

Ask Riyaz Sir