IGCSE MATHEMATICS 0580 · CHAPTER 5 NOTES · ~18 MIN READ

Sequences, functions & graphs

Spot patterns and write the nth term, use function machines, sketch and read every graph shape, and meet the gradient of a curve and differentiation, all with clear pictures.

nth term | Functions & inverse | Straight-line & curve graphs | Gradient of curves | Differentiation

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No tag = **Core** (everyone) EXTENDED Extended only highlight = comes up again and again

Test yourself at the end

IN 30 SECONDS

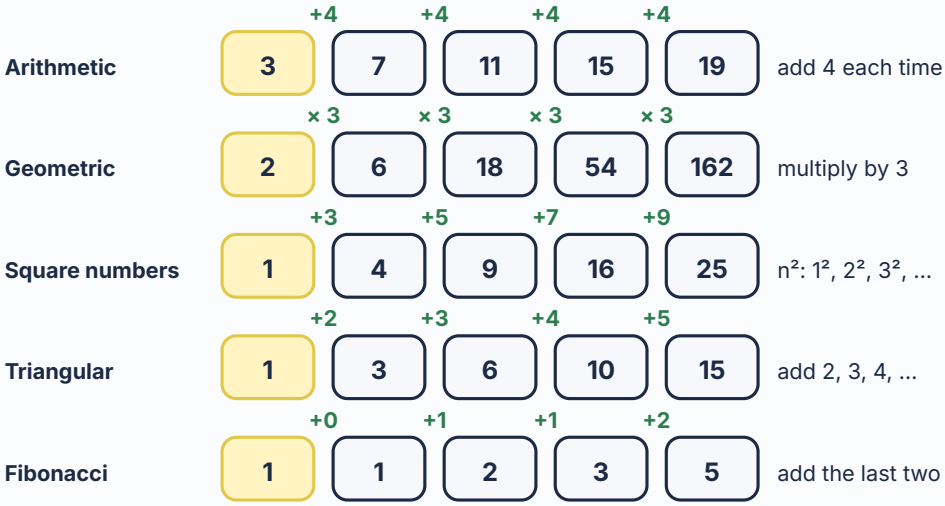
- A **sequence** follows a rule. For a **linear** sequence the **nth term** is **dn + c**, where d is the **common difference**. A **function** is a machine: **inverse** = undo it, **composite** = do one then the other.
- **Graphs have shapes**: straight lines, U-shaped parabolas, S-shaped cubics and the two-branch reciprocal. **Gradient** on a distance-time graph is **speed**, and on a speed-time graph it is **acceleration**. The **area** under a speed-time graph is **distance**.
- **Extended:** quadratic sequences, function notation, **tangents** and **differentiation** to find gradients and turning points.

BY THE END YOU CAN

- Continue sequences and describe the term-to-term rule.
- Find the nth term of a linear sequence and test whether a number is in it.
- Recognise and sketch straight-line, quadratic, cubic and reciprocal graphs, and plot graphs from tables of values.
- Solve equations by reading intersections from graphs.
- Draw and interpret distance-time and speed-time graphs.
- Find the nth term of quadratic and exponential sequences. EXT
- Use function notation, composite and inverse functions. EXT
- Estimate the gradient of a curve with a tangent, and differentiate powers of x. EXT
- Find turning points and apply differentiation to problems. EXT

1 Sequences

Types of sequence



Find the difference or the ratio between neighbouring terms. That tells you which kind of sequence you have.

Term — one number in a sequence. The first term is T(1), the second T(2), and the nth term is T(n).

Term-to-term rule — how to get from one term to the next, for example "add 4".

nth term — a formula that gives **any** term directly from its position n.

- **Arithmetic (linear): same difference** each time. **Geometric: same multiplier**. **Square numbers:** 1, 4, 9, 16, ... **Cube numbers:** 1, 8, 27, 64, ... **Triangular:** 1, 3, 6, 10, ...
- **Fibonacci-type:** each term is the **sum of the two before**: 1, 1, 2, 3, 5, 8, 13, ...

WORKED EXAMPLE

Write the next two terms of 2, 6, 18, 54, ...

1. 6 ÷ 2 = 3, 18 ÷ 6 = 3, so the rule is **× 3**
2. 54 × 3 = **162**
3. **162 × 3 = 486**

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nth term of a linear sequence



the difference is 3, so start with 3n, then add 2: nth term = 3n + 2

The difference tells you the number in front of n. Then adjust with a constant.

$$\text{nth term} = dn + c$$

d = common difference. c = (first term) − d

- **Method:** (1) find the **common difference** d; (2) write **dn**; (3) compare with the sequence and **add or subtract** the number that fixes it.
- **Is a number in the sequence?** Set the nth term equal to the number and solve for n. If n is a **whole number**, it is a term.

WORKED EXAMPLE

Find the nth term of 20, 17, 14, 11, ...

1. common difference d = **−3**, so start with **−3n**
2. n = 1: −3(1) = −3, but the term is 20, so add **23**
3. nth term = **−3n + 23** (or 23 − 3n)
4. **check n = 4: −12 + 23 = 11 ✓**

WORKED EXAMPLE

Find the 50th term of 4n − 1, and decide whether 150 is a term.

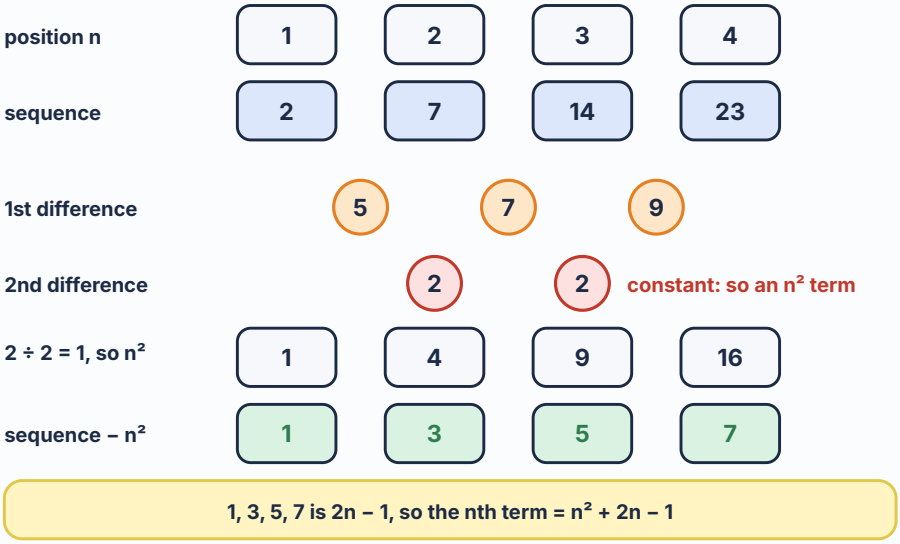
1. 50th term: 4 × 50 − 1 = **199**
2. set 4n − 1 = 150, so 4n = 151
3. n = 37.75, **not a whole number**
4. **150 is not a term**

COMMON MISTAKE

Writing the nth term as 3n + 5 for the sequence 5, 8, 11. Check n = 1: 3 + 5 = 8, **not 5**. The constant is **first term − d**: 5 − 3 = **2**.

Quadratic and exponential sequences

EXTENDED



A constant second difference means the nth term has n^2 in it. The number in front of n^2 is half the second difference.

- EXT **Quadratic:** the **second difference** is constant. The **coefficient of n^2** is **half** the second difference. Subtract the n^2 terms, then find the **linear** nth term of what is left.
- EXT **Exponential (geometric):** nth term = **$a \times r^{n-1}$**, where a is the first term and r the common ratio. 3, 6, 12, 24, … is $3 \times 2^{n-1}$.

WORKED EXAMPLE

EXT **Find the nth term of 4, 10, 18, 28, …**

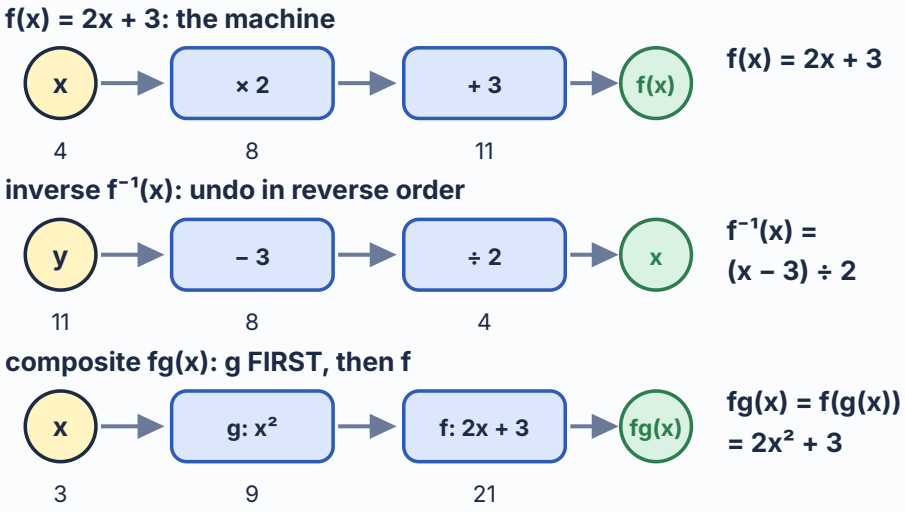
1. first differences: 6, 8, 10
2. second differences: 2, 2, so a = 2 ÷ 2 = **1**: n^2
3. n^2 gives 1, 4, 9, 16
4. sequence – n^2: 3, 6, 9, 12 = **3n**
5. nth term = **$n^2 + 3n$**
6. **check n = 3: 9 + 9 = 18 ✓**

WORKED EXAMPLE

EXT **Find the nth term of 2, 6, 18, 54, …**

1. ratio = 3 and first term = 2
2. nth term = **$2 \times 3^{n-1}$**
3. **check n = 4: $2 \times 3^3 = 54$ ✓**

Function notation, composite and inverse



A function is a machine: input x, output f(x). Composite functions apply two machines in order, and the inverse undoes the machine.

Function — a rule that gives **exactly one output** for every input. f(x) = 2x + 3 means "double x, then add 3".

Composite function — fg(x) means **g first, then f**: fg(x) = f(g(x)).

Inverse function — f^{-1} **undoes** f. It takes the output back to the input.

- EXT **f(4)** means "put x = 4 into f": f(4) = 2(4) + 3 = **11**.
- EXT **To find f^{-1}:** write y = f(x), **make x the subject**, then swap y back to x. (See rearranging formulae.)

WORKED EXAMPLE

EXT **f(x) = 3x − 2. Find f(4) and solve f(x) = 13.**

1. f(4) = 3(4) − 2 = **10**
2. 3x − 2 = 13
3. **3x = 15, so x = 5**

WORKED EXAMPLE

EXT **f(x) = 2x + 3 and g(x) = x^2. Find fg(x) and gf(x).**

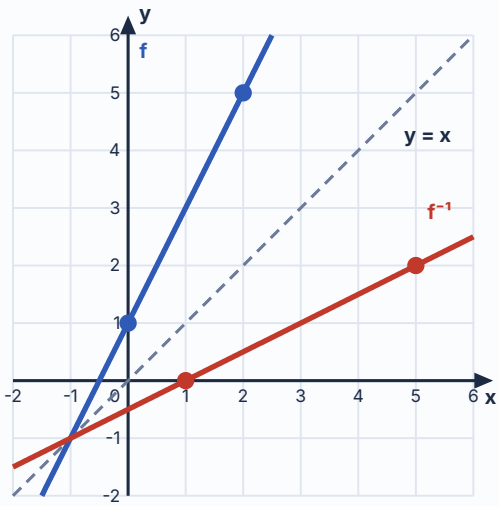
1. fg(x) = f(x^2) = **$2x^2 + 3$** (square first, then double and add 3)
2. gf(x) = g(2x + 3) = **$(2x + 3)^2$**
3. = $4x^2 + 12x + 9$
4. **the order matters: fg ≠ gf**

WORKED EXAMPLE

EXT $f(x) = \frac{x + 5}{3}$. Find $f^{-1}(x)$.

1. $y = \frac{x + 5}{3}$
2. multiply by 3: $3y = x + 5$
3. $x = 3y - 5$
4. **swap: $f^{-1}(x) = 3x - 5$**

The graph of an inverse



f(x) = 2x + 1

$f^{-1}(x) = (x - 1) \div 2$

The graph of f^{-1} is the reflection of f in the line y = x.
Swap the coordinates:
(0, 1) on f ↔ (1, 0) on f^{-1}
(2, 5) on f ↔ (5, 2) on f^{-1}

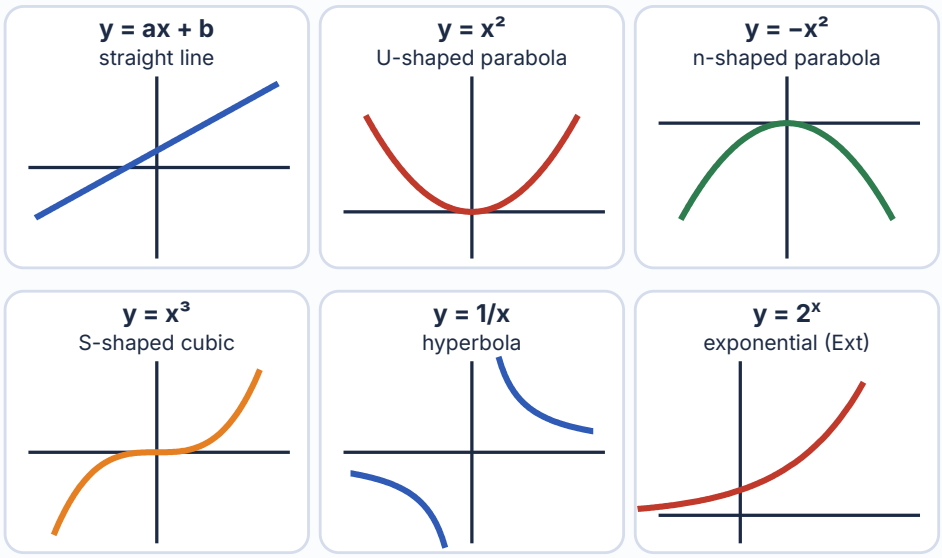
The graph of the inverse is the mirror image of the original in the line y = x. Every point (a, b) becomes (b, a).

- EXT **Reflect** the graph of f in the line **y = x** to get the graph of f^{-1}. **Swap the coordinates** of every point.

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3 Graphs

Shapes of graphs



Recognise the shape from the equation. A positive x^2 gives a U, a negative x^2 gives an n.

Equation	Shape	Key facts
$y = ax + b$	straight line	gradient a, y-intercept b
$y = ax^2 + bx + c$	parabola	U if $a > 0$, n if $a < 0$
$y = ax^3 + \ldots$	cubic	S-shape, up to two turning points
$y = k/x$	hyperbola	two branches, never touches the axes
$y = a^x$ (Extended)	exponential	passes (0, 1), never below the x-axis

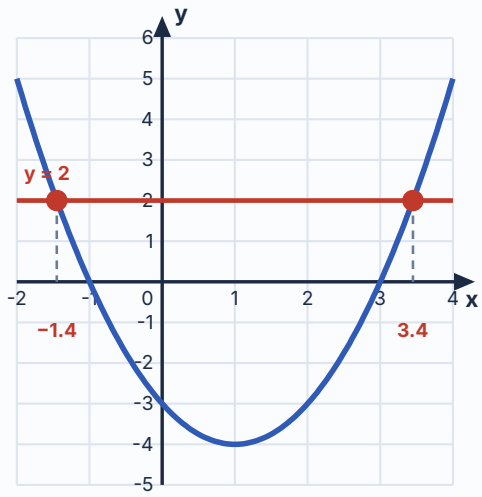
- **Plot from a table:** substitute each x to get y, plot the points, and join with a **smooth curve** (not straight ruler lines).
- **Check the shape:** a parabola is **symmetrical**, so y-values repeat on either side of the turning point.

WORKED EXAMPLE

Complete a table for $y = x^2 - 3x$, for $x = -1$ to 4, and find the turning point.

1. $x = -1$: $1 + 3 = \mathbf{4}$
2. $x = 0$: $\mathbf{0}$
3. $x = 1$: $1 - 3 = \mathbf{-2}$
4. $x = 2$: $4 - 6 = \mathbf{-2}$
5. $x = 3$: $9 - 9 = \mathbf{0}$
6. $x = 4$: $16 - 12 = \mathbf{4}$
7. **symmetrical about $x = 1.5$, so the turning point is at $(1.5, -2.25)$**

Solving equations from graphs



Solve $x^2 - 2x - 5 = 0$

same as $x^2 - 2x - 3 = 2$
1. draw $y = x^2 - 2x - 3$
2. draw the line $y = 2$
3. read the x-values where they cross:

$x \approx -1.4$ or $x \approx 3.4$

To solve f(x) = a number, draw the horizontal line y = that number and read the x-values where it meets the curve.

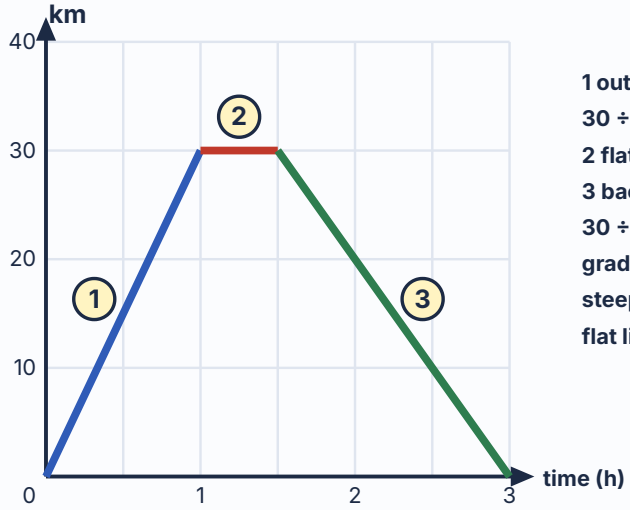
- **Solve f(x) = k:** draw the line **y = k**. The **x-coordinates** of the crossing points are the solutions.
- **Solve f(x) = g(x):** draw **both** graphs. The x-coordinates of the crossings are the solutions. For example, $y = x^2 - 2x - 3$ meets the line $y = x + 1$ at **$x = -1$ and $x = 4$**.
- Read to the **accuracy of your graph**: "$x \approx -1.4$" means about.

WORKED EXAMPLE

Use the graph of $y = x^2 - 2x - 3$ to solve $x^2 - 2x - 3 = 2$.

1. draw the horizontal line **y = 2**
2. it crosses the curve at two points
3. read the x-values: **$x \approx -1.4$ and $x \approx 3.4$**
4. **(this solves $x^2 - 2x - 5 = 0$)**

Distance–time graphs



1 out: 30 km in 1 h
30 ÷ 1 = 30 km/h
2 flat: stopped for 0.5 h
3 back: 30 km in 1.5 h
30 ÷ 1.5 = 20 km/h
gradient = speed
steeper line = faster
flat line = not moving

Gradient means speed. A steeper line is a faster journey, and a flat line means stopped.

- **Gradient = distance ÷ time = speed. Flat line:** not moving. **Return journey:** the line goes **down**.
- **Average speed** = **total distance ÷ total time** (including any stops).

WORKED EXAMPLE

Use the graph. Find the speed on the return journey and the average speed for the whole trip.

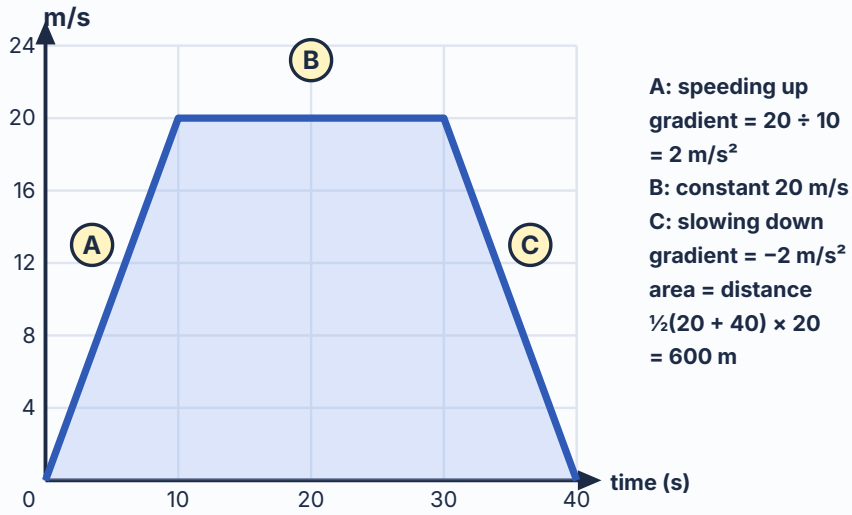
1. return: 30 km in 1.5 h
2. 30 ÷ 1.5 = **20 km/h**
3. total distance = 30 + 30 = 60 km, total time = 3 h
4. **average speed = 60 ÷ 3 = 20 km/h**

WORKED EXAMPLE

A cyclist rides 12 km in 40 minutes. Find the speed in km/h.

1. 40 min = $\frac{40}{60}$ h = $\frac{2}{3}$ h
2. $12 \div \frac{2}{3} = 12 \times \frac{3}{2}$
3. **= 18 km/h**

Speed-time graphs



Gradient means acceleration. The area under the graph is the distance travelled.

$$\text{acceleration} = \frac{\text{change in speed}}{\text{time taken}}$$

= gradient of a speed-time graph. Negative = deceleration (slowing down)

- **Gradient = acceleration** (m/s²). **Area under the graph = distance.** A **flat** line means constant speed.
- EXT For a **curved** graph: the gradient at a point is the gradient of a **tangent**; the area can be **estimated with trapezia**.

WORKED EXAMPLE

Use the graph to find the acceleration in the first 10 s and the total distance.

1. acceleration = $\frac{20}{10}$ = **2 m/s²**
2. total distance = area of the trapezium
3. = ½ × (20 + 40) × 20
4. **= 600 m**

WORKED EXAMPLE

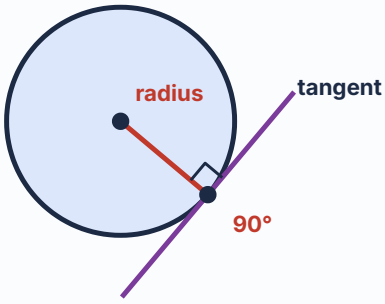
EXT **Speeds are 0, 6, 10 and 12 m/s at times 0, 2, 4 and 6 s. Estimate the distance using trapezia.**

1. strip 1: ½ × 2 × (0 + 6) = **6**
2. strip 2: ½ × 2 × (6 + 10) = **16**
3. strip 3: ½ × 2 × (10 + 12) = **22**
4. **total: 6 + 16 + 22 = 44 m**

4 Gradient of curves and differentiation EXT

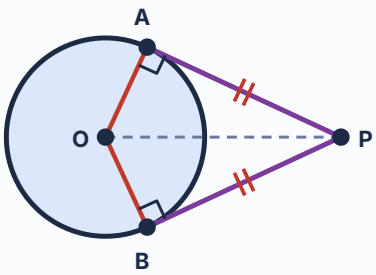
The gradient of a curve

Tangent and radius



a tangent meets the radius at exactly 90°

Two tangents from a point



two tangents from a point are equal: PA = PB
(triangle OAP and OBP are congruent: RHS)

The gradient of a curve changes from point to point. At a point it equals the gradient of the tangent there.

Tangent — a straight line that **touches** a curve at one point and has the **same direction** as the curve there.

- EXT **To estimate:** draw a tangent with a ruler, make a **right-angled triangle**, then **gradient = rise ÷ run**.
- EXT A **positive** gradient means the curve is **rising**; **negative**, **falling**; **zero**, flat (a turning point).

Differentiation

$y = ax^n \rightarrow dy/dx = n \times a \times x^{n-1}$

(a and n are numbers, x is the letter)

1. multiply by the power	2. reduce the power by 1

$y = x^4$ $dy/dx = 4x^3$	$y = 3x^2$ $dy/dx = 6x$	$y = 5x$ $dy/dx = 5$	$y = 7$ $dy/dx = 0$

$y = 2x^3 - 6x + 1$ $dy/dx = 6x^2 - 6$	$y = 1/x = x^{-1}$ $dy/dx = -x^{-2} = -1/x^2$	$y = \sqrt{x} = x^{½}$ $dy/dx = ½x^{-½}$

last row: Extended

Multiply by the power, then reduce the power by one. A number on its own differentiates to zero.

Derivative — the **gradient function**, dy/dx. It gives the gradient of the curve at **any** x.

- EXT **Differentiate each term separately.** Rewrite roots and fractions as powers first: $\frac{1}{x} = x^{-1}$ and $\sqrt{x} = x^{½}$.

WORKED EXAMPLE

EXT **Differentiate $y = 5x^3 - 2x^2 + 7x - 4$.**

1. $5x^3 \rightarrow 5 \times 3 \times x^2 =$ **$15x^2$**
2. $-2x^2 \rightarrow -2 \times 2 \times x =$ **$-4x$**
3. $7x \rightarrow$ **7**
4. $-4 \rightarrow$ **0**
5. **$dy/dx = 15x^2 - 4x + 7$**

WORKED EXAMPLE

EXT **Find the gradient of $y = x^2 + 3x$ at $x = 2$.**

1. $dy/dx =$ **$2x + 3$**
2. **when $x = 2$: $2(2) + 3 = 7$**

WORKED EXAMPLE

EXT **Differentiate $y = \frac{4}{x^2}$.**

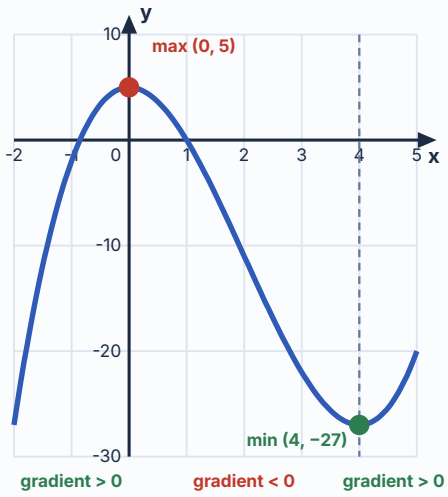
1. rewrite: $y = 4x^{-2}$
2. multiply by the power: $4 \times (-2) = -8$
3. reduce the power: x^{-3}
4. **$dy/dx = -8x^{-3} = -\frac{8}{x^3}$**

WORKED EXAMPLE

EXT **Find the equation of the tangent to $y = x^2$ at the point (2, 4).**

1. $dy/dx = 2x$, so the gradient at $x = 2$ is **4**
2. $y - 4 = 4(x - 2)$
3. $y - 4 = 4x - 8$
4. **$y = 4x - 4$**

Turning points



$y = x^3 - 6x^2 + 5$

$dy/dx = 3x^2 - 12x$

Turning points: $dy/dx = 0$
$3x(x - 4) = 0$
$x = 0$ or $x = 4$
$x = 0$: $y = 5$ (maximum)
$x = 4$: $y = -27$ (minimum)

At a turning point the gradient is zero. Solve $dy/dx = 0$, then find y, then decide whether it is a maximum or minimum.

- EXT **Method:** (1) find dy/dx; (2) **set dy/dx = 0** and solve; (3) find **y** for each x; (4) decide **maximum or minimum** using the **gradient either side**, or the **second derivative**: positive → minimum, negative → maximum.

WORKED EXAMPLE

EXT **Find and classify the turning points of $y = x^3 - 6x^2 + 5$.**

1. $dy/dx = 3x^2 - 12x = 3x(x - 4)$
2. $dy/dx = 0$: **$x = 0$ or $x = 4$**
3. $y(0) = 5$, $y(4) = 64 - 96 + 5 = -27$
4. second derivative $= 6x - 12$: at $x = 0$ it is -12 (negative), so **(0, 5) is a maximum**
5. **at $x = 4$ it is $+12$ (positive), so $(4, -27)$ is a minimum**

WORKED EXAMPLE

EXT **A farmer has 40 m of fencing for a rectangle with width x. Find the maximum area.**

1. length $= 20 - x$, so area $A = x(20 - x) = 20x - x^2$
2. $dA/dx = 20 - 2x = 0$, so **$x = 10$**
3. **$A = 10 \times 10 = 100\ m^2$ (a maximum, because the x^2 term is negative)**

Test yourself — cover the answers first

QUICK CHECK

1. Write the next two terms of 2, 6, 18, 54.
2. Find the nth term of 7, 11, 15, 19.
3. Find the nth term of 20, 17, 14, 11.
4. Is 100 a term of the sequence $3n + 1$?
5. What is the next term of 1, 1, 2, 3, 5, 8?
6. Complete $y = x^2 - 3x$ when $x = 2$.
7. What does a flat line on a distance-time graph mean?
8. What does the area under a speed-time graph represent?
9. EXT Find the nth term of 3, 6, 11, 18.
10. EXT $f(x) = 5x - 1$. Find $f(3)$.
11. EXT $f(x) = 2x + 1$ and $g(x) = x - 4$. Find $fg(x)$.
12. EXT Find the inverse of $f(x) = \frac{x + 5}{3}$.
13. EXT Differentiate $y = 3x^4 - 5x$.
14. EXT Differentiate $y = \frac{6}{x^2}$.
15. EXT Find the gradient of $y = x^2 + 3x$ at $x = 2$.
16. EXT Find the turning point of $y = x^2 - 4x + 1$.

EXAM-STYLE

17. A sequence begins 5, 9, 13, 17. Find the nth term and decide whether 99 is a term. [3]
18. Complete the table for $y = x^2 - 2x - 1$ for $x = -1, 0, 1, 2, 3$. [2]
19. A car accelerates from rest to 24 m/s in 8 s, travels at 24 m/s for 20 s, then brakes to rest in 6 s. Find the initial acceleration and the total distance. [4]
20. EXT $f(x) = 3x + 2$ and $g(x) = x^2$. Find $fg(2)$ and $f^{-1}(x)$. [4]
21. EXT Find the nth term of 2, 5, 10, 17, 26. [3]
22. EXT Find the coordinates of the turning points of $y = x^3 - 3x + 2$ and decide whether each is a maximum or minimum. [5]
23. EXT Find the equation of the tangent to $y = x^2$ at the point where $x = 3$. [3]
24. EXT Speeds are 0, 6, 10 and 12 m/s at 0, 2, 4 and 6 s. Estimate the distance using trapezia. [3]

Show answers

1. 162, 486
2. $4n + 3$
3. $23 - 3n$
4. Yes ($n = 33$)
5. 13
6. -2
7. the object is stationary
8. the distance travelled
9. $n^2 + 2$
10. 14
11. $2x - 7$
12. $f^{-1}(x) = 3x - 5$
13. $12x^3 - 5$
14. $-12x^{-3}$
15. 7
16. (2, −3), a minimum
17. $d = 4$, so nth term $= 4n + 1$
    $4n + 1 = 99$ gives $n = 24.5$
    not a whole number, so 99 is not a term
18. $x = -1$: $1 + 2 - 1 = 2$
    $x = 0$: -1
    $x = 1$: $1 - 2 - 1 = -2$
    $x = 2$: $4 - 4 - 1 = -1$
    $x = 3$: $9 - 6 - 1 = 2$
19. acceleration $= 24 \div 8 = 3$ m/s²
    total time = 34 s, parallel sides 34 and 20
    area $= \frac{1}{2} \times (34 + 20) \times 24 = 648$ m
20. $g(2) = 4$, then $f(4) = 14$
    $y = 3x + 2$, so $x = (y - 2) \div 3$
    $f^{-1}(x) = (x - 2) \div 3$
21. first differences 3, 5, 7, 9, second difference 2, so n^2
    $2 - 1 = 1$, $5 - 4 = 1$, $10 - 9 = 1$, ...
    nth term $= n^2 + 1$
22. $dy/dx = 3x^2 - 3 = 0$, so $x = 1$ or $x = -1$
    $y(1) = 0$, $y(-1) = 4$
    second derivative $6x$: at $x = 1$ it is 6 (positive): minimum at (1, 0)
    at $x = -1$ it is -6 (negative): maximum at (−1, 4)
23. $dy/dx = 2x = 6$ at $x = 3$
    $y = 9$ at $x = 3$
    $y - 9 = 6(x - 3)$, so $y = 6x - 9$

24. ½ × 2 × (0 + 6) = 6
½ × 2 × (6 + 10) = 16
½ × 2 × (10 + 12) = 22
total = 44 m

Stuck on a question?

Send it to me on WhatsApp. I read every message.

Want regular help? Send this to your parent →

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